

Radon transforms for quasi-equivariant \mathcal{D} -modules on generalized flag manifolds

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Abstract

In this paper we deal with Radon transforms for generalized flag manifolds in the framework of quasi-equivariant \mathcal{D} -modules. We shall follow the method employed by Baston-Eastwood and analyze the Radon transform using the Bernstein-Gelfand-Gelfand resolution and the Borel-Weil-Bott theorem. We shall determine the transform completely on the level of the Grothendieck groups. Moreover, we point out a vanishing criterion and give a sufficient condition in order that a \mathcal{D} -module associated to an equivariant locally free \mathcal{O} -module is transformed into an object of the same type. The case of maximal parabolic subgroups of classical simple groups is studied in detail.

Introduction

Let G be a reductive algebraic group over \mathbb{C} , P and Q two parabolic subgroups containing the same Borel subgroup of G . Let $X = G/P$, $Y = G/Q$, and let S be the unique closed G -orbit in $X \times Y$ for the diagonal action. Then we can identify S with $G/P \cap Q$. The natural correspondence

$$X \xleftarrow{f} S \xrightarrow{g} Y,$$

where f and g are the restriction to S of the projections of $X \times Y$ on X and Y , induces an integral transform from X to Y which generalizes the classical Radon-Penrose transform. This subject has been investigated intensively both in the complex and real domains (see e.g. Baston-Eastwood [1], D'Agnolo-Schapira [5], Kakehi [6], Marastoni [10], Oshima [12], Sekiguchi [14], Tanisaki [15]).

Our aim is to study this transform in the framework of quasi- G -equivariant \mathcal{D} -modules (see Kashiwara [7]), i.e. the functor

$$R : \mathbf{D}_G^b(\mathcal{D}_X) \rightarrow \mathbf{D}_G^b(\mathcal{D}_Y), \quad R(\mathcal{M}) = g_* f^{-1} \mathcal{M}, \quad (0.1)$$

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where $\mathbf{D}_G^b(\mathcal{D})$ denotes the derived category of quasi- G -equivariant \mathcal{D} -modules with bounded cohomologies, and \underline{g}_* and \underline{f}^{-1} are the operations of direct image (integration) and inverse image (pull-back) for \mathcal{D} -modules. More precisely, we consider a \mathcal{D}_X -module of type $\mathcal{M} = \mathcal{D}\mathcal{L} = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{L}$, where \mathcal{L} is an irreducible G -equivariant locally free \mathcal{O}_X -module. In this case it is easily seen that

$$H^p(R(\mathcal{D}\mathcal{L})) = 0 \quad \text{for any } p < 0 \quad (0.2)$$

(see Lemma 1.4 below). Note that the Grothendieck group of the category of quasi- G -equivariant \mathcal{D}_X -modules of finite length is spanned by elements corresponding to the objects of the form $\mathcal{D}\mathcal{L}$.

As in Baston-Eastwood [1] our analysis relies on the Bernstein-Gelfand-Gelfand resolution and the Borel-Weil-Bott theorem. Using the Bernstein-Gelfand-Gelfand resolution in the parabolic setting (see Bernstein-Gelfand-Gelfand [2], Lepowsky [9], Rocha-Caridi [13]) we obtain a resolution of the quasi- G -equivariant \mathcal{D}_S -module $\underline{f}^{-1}(\mathcal{D}\mathcal{L})$ of the form:

$$0 \rightarrow \bigoplus_{k=1}^{r_n} \mathcal{D}\mathcal{L}_{nk} \rightarrow \cdots \rightarrow \bigoplus_{k=1}^{r_0} \mathcal{D}\mathcal{L}_{0k} \rightarrow \underline{f}^{-1}(\mathcal{D}\mathcal{L}) \rightarrow 0, \quad (0.3)$$

where \mathcal{L}_{ik} are irreducible G -equivariant locally free \mathcal{O}_S -modules (see § 2.2 for the explicit description of \mathcal{L}_{ik}). Then we have

$$\underline{g}_*(\mathcal{D}\mathcal{L}_{ik}) = \mathcal{D}_Y \otimes_{\mathcal{O}_Y} Rg_*(\mathcal{L}_{ik} \otimes_{\mathcal{O}_S} \Omega_g)$$

by the definition of \underline{g}_* , where Ω_g denotes the sheaf of relative differential forms with maximal degree along the fibers of g . Moreover, the Borel-Weil-Bott theorem tells us the structure of $Rg_*(\mathcal{L}_{ik} \otimes_{\mathcal{O}_S} \Omega_g)$. In particular, we have either $Rg_*(\mathcal{L}_{ik} \otimes_{\mathcal{O}_S} \Omega_g) = 0$ or there exist a non-negative integer m_{ik} and an irreducible G -equivariant \mathcal{O}_Y -module \mathcal{L}'_{ik} such that $Rg_*(\mathcal{L}_{ik} \otimes_{\mathcal{O}_S} \Omega_g) = \mathcal{L}'_{ik}[-m_{ik}]$. Thus setting

$$\mathcal{I} = \{(i, k) ; 0 \leq i \leq n, 1 \leq k \leq r_i, Rg_*(\mathcal{L}_{ik} \otimes_{\mathcal{O}_S} \Omega_g) \neq 0\},$$

we have

$$\underline{g}_*(\mathcal{D}\mathcal{L}_{ik}) = \begin{cases} \mathcal{D}\mathcal{L}'_{ik}[-m_{ik}] & ((i, k) \in \mathcal{I}), \\ 0 & ((i, k) \notin \mathcal{I}) \end{cases} \quad (0.4)$$

(see §2.2 below for concrete descriptions of \mathcal{I} and \mathcal{L}_{ik}, m_{ik} for $(i, k) \in \mathcal{I}$).

Then we can study the structure of $R(\mathcal{D}\mathcal{L}) = \underline{g}_* \underline{f}^{-1}(\mathcal{D}\mathcal{L})$ using (0.2), (0.3) and (0.4). For example we have the following result.

Theorem 0.1. *Let the notation be as above.*

(i) We have

$$\sum_p (-1)^p [H^p(R(\mathcal{DL}))] = \sum_{(i,k) \in \mathcal{I}} (-1)^{i-m_{ik}} [\mathcal{DL}'_{ik}]$$

in the Grothendieck group of the category of quasi- G -equivariant \mathcal{D}_Y -modules.

(ii) If $\mathcal{I} = \emptyset$, then $R(\mathcal{DL}) = 0$.

(iii) If \mathcal{I} consists of a single element (i, k) , then $R(\mathcal{DL}) = \mathcal{DL}'_{ik}[i - m_{ik}]$.

(iv) If $i \geq m_{ik}$ for any $(i, k) \in \mathcal{I}$, then we have $H^p(R(\mathcal{DL})) = 0$ unless $p = 0$.

(v) If $i > m_{ik}$ for any $(i, k) \in \mathcal{I}$ with $i > 0$ and if $m_{01} = 0$, then there exists an epimorphism $\mathcal{D}_Y \mathcal{L}'_{01} \rightarrow H^0(R(\mathcal{DL}))$ (note that $r_0 = 1$).

Assume that \mathcal{L} is invertible and that there exists a G -equivariant invertible \mathcal{O}_Y -module \mathcal{L}' satisfying $f^* \mathcal{L} \otimes_{\mathcal{O}_S} \Omega_g = g^* \mathcal{L}'$. We call such a pair $(\mathcal{L}, \mathcal{L}')$ an extremal case for the correspondence (if $P \cup Q$ generates the group G and if G is semisimple, then there exists a unique extremal case). In this case there exists a natural nontrivial \mathcal{D}_Y -linear morphism

$$\Phi : \mathcal{DL}' \rightarrow H^0(R(\mathcal{DL})). \quad (0.5)$$

Theorem 0.2. *Let $(\mathcal{L}, \mathcal{L}')$ be an extremal case.*

(i) *We have $H^p(R(\mathcal{DL})) = 0$ for any $p \neq 0$ if and only if $i \geq m_{ik}$ for any $(i, k) \in \mathcal{I}$.*

(ii) *Assume that $H^p(R(\mathcal{DL})) = 0$ for any $p \neq 0$. Then Φ is an epimorphism if and only if $i > m_{ik}$ for any $(i, k) \in \mathcal{I}$ with $i > 0$.*

(iii) *Assume that $H^p(R(\mathcal{DL})) = 0$ for any $p \neq 0$. Then Φ is an isomorphism if and only if \mathcal{I} consists of a single element $(0, 1)$.*

We do not know an example of an extremal case $(\mathcal{L}, \mathcal{L}')$ such that $H^p(R(\mathcal{DL})) \neq 0$ for some $p \neq 0$. We have checked that $H^p(R(\mathcal{DL})) = 0$ for any $p \neq 0$ by a case-by-case analysis when G is a classical simple group, P, Q are maximal parabolic subgroups and $(\mathcal{L}, \mathcal{L}')$ is the extremal case. In general the morphism Φ for an extremal case $(\mathcal{L}, \mathcal{L}')$ is not necessarily an epimorphism nor a monomorphism. It would be an interesting problem to determine the kernel and the cokernel of Φ .

The transform of a \mathcal{D} -module, a problem of analytic nature, is not sufficient to cover the general problem of integral geometry. In order to do this, one should couple the transforms in the frameworks of \mathcal{D} -modules and sheaves. This is better described in the adjunction formulas (see D'Agnolo-Schapira [5]), and we shall briefly discuss this point with an example in the case of $G = SL_{n+1}(\mathbb{C})$.

We would like to thank M. Kashiwara for useful conversation on quasi-equivariant \mathcal{D} -modules.

1 Preliminaries on \mathcal{D} -modules

1.1 Functors for \mathcal{D} -modules

Let Z be an algebraic manifold (smooth algebraic variety) over \mathbb{C} . We denote by \mathcal{O}_Z the structure sheaf, by Ω_Z the invertible \mathcal{O}_Z -module of differential forms of maximal degree, and by \mathcal{D}_Z the sheaf of differential operators. In this paper an \mathcal{O}_Z -module means a quasi-coherent \mathcal{O}_Z -module and a \mathcal{D}_Z -module means a left \mathcal{D}_Z -module which is quasi-coherent over \mathcal{O}_Z . We denote by $\text{Mod}(\mathcal{D}_Z)$ the category of \mathcal{D}_Z -modules and by $\mathbf{D}^b(\mathcal{D}_Z)$ the derived category of $\text{Mod}(\mathcal{D}_Z)$ whose objects have bounded cohomology.

If $f : Z \rightarrow Z'$ is a morphism, we set

$$\Omega_f = \Omega_{Z/Z'} = \Omega_Z \otimes_{f^{-1}\mathcal{O}_{Z'}} f^{-1}\Omega_{Z'}^{\otimes -1};$$

and, for an $\mathcal{O}_{Z'}$ -module \mathcal{L}' , we set

$$f^*\mathcal{L}' = \mathcal{O}_Z \otimes_{f^{-1}\mathcal{O}_{Z'}} f^{-1}\mathcal{L}', \quad Lf^*\mathcal{L}' = \mathcal{O}_Z \otimes_{f^{-1}\mathcal{O}_{Z'}}^L f^{-1}\mathcal{L}'.$$

We denote by \underline{f}_* and \underline{f}^{-1} the direct and inverse image for left \mathcal{D} -modules:

$$\begin{aligned} \underline{f}_* : \mathbf{D}^b(\mathcal{D}_Z) &\rightarrow \mathbf{D}^b(\mathcal{D}_{Z'}), & \underline{f}_*\mathcal{M} &= Rf_*(\mathcal{D}_{Z' \leftarrow Z} \otimes_{\mathcal{D}_Z}^L \mathcal{M}), \\ \underline{f}^{-1} : \mathbf{D}^b(\mathcal{D}_{Z'}) &\rightarrow \mathbf{D}^b(\mathcal{D}_Z), & \underline{f}^{-1}\mathcal{M}' &= \mathcal{D}_{Z \rightarrow Z'} \otimes_{f^{-1}\mathcal{D}_{Z'}}^L f^{-1}\mathcal{M}', \end{aligned}$$

where a $(\mathcal{D}_Z, f^{-1}\mathcal{D}_{Z'})$ -bimodule $\mathcal{D}_{Z \rightarrow Z'}$ and an $(f^{-1}\mathcal{D}_{Z'}, \mathcal{D}_Z)$ -bimodule $\mathcal{D}_{Z' \leftarrow Z}$ are defined by

$$\mathcal{D}_{Z \rightarrow Z'} = \mathcal{O}_Z \otimes_{f^{-1}\mathcal{O}_{Z'}} f^{-1}\mathcal{D}_{Z'}, \quad \mathcal{D}_{Z' \leftarrow Z} = \Omega_Z \otimes_{\mathcal{O}_Z} \mathcal{D}_{Z \rightarrow Z'} \otimes_{f^{-1}\mathcal{O}_{Z'}} f^{-1}\Omega_{Z'}^{\otimes -1}.$$

Note that for a $\mathcal{D}_{Z'}$ -module \mathcal{M} we have $\underline{f}^{-1}\mathcal{M} \simeq Lf^*\mathcal{M}$ as a complex of \mathcal{O}_Z -modules. Note also that we have canonical morphisms $\mathcal{O}_Z \rightarrow \mathcal{D}_{Z \rightarrow Z'}$ and $\Omega_f \rightarrow \mathcal{D}_{Z' \leftarrow Z}$ of \mathcal{O}_Z -modules.

The following result is well-known and easy to prove.

Lemma 1.1. *Let $f_1 : Z \rightarrow X_1$ and $f_2 : Z \rightarrow X_2$ be morphisms of algebraic manifolds.*

(i) *We have*

$$\mathcal{D}_{X_2 \leftarrow Z} \otimes_{\mathcal{D}_Z}^L \mathcal{D}_{Z \rightarrow X_1} \xrightarrow{\sim} f_1^{-1}\Omega_{X_1} \otimes_{f_1^{-1}\mathcal{O}_{X_1}}^L (\mathcal{D}_{X_1 \times X_2 \leftarrow Z} \otimes_{\mathcal{D}_Z}^L \mathcal{O}_Z).$$

(ii) *Assume that $Z \rightarrow X_1 \times X_2$ is an embedding. Then we have*

$$\mathcal{D}_{X_2 \leftarrow Z} \otimes_{\mathcal{D}_Z}^L \mathcal{D}_{Z \rightarrow X_1} = \mathcal{D}_{X_2 \leftarrow Z} \otimes_{\mathcal{D}_Z} \mathcal{D}_{Z \rightarrow X_1},$$

and the canonical morphism $\Omega_{f_2} \rightarrow \mathcal{D}_{X_2 \leftarrow Z} \otimes_{\mathcal{D}_Z} \mathcal{D}_{Z \rightarrow X_1}$ of $(f_2^{-1}\mathcal{O}_{X_2}, f_1^{-1}\mathcal{O}_{X_1})$ -bimodules is a monomorphism.

For a locally free \mathcal{O}_Z -module \mathcal{L} , we set

$$\mathcal{DL} = \mathcal{D}_Z \otimes_{\mathcal{O}_Z} \mathcal{L},$$

and for a closed submanifold Z of an algebraic manifold X we define a \mathcal{D}_X -module $\mathcal{B}_{Z|X}$ supported on Z by

$$\mathcal{B}_{Z|X} = H_{[Z]}^d(\mathcal{O}_X) = i_* \mathcal{O}_Z,$$

where $d = \text{codim}_X Z$ and $i : Z \rightarrow X$ denotes the embedding.

1.2 Radon transforms

Let X and Y be algebraic manifolds over \mathbb{C} , and denote by q_1 and q_2 the projections of $X \times Y$ onto X and Y respectively. Let S be a locally closed submanifold of $X \times Y$ and let $i : S \rightarrow X \times Y$ be the embedding. The geometric correspondence

$$X \xleftarrow{f} S \xrightarrow{g} Y \quad (1.1)$$

where f and g are the restrictions of q_1 and q_2 , induces a functor

$$R : \mathbf{D}^b(\mathcal{D}_X) \rightarrow \mathbf{D}^b(\mathcal{D}_Y), \quad R(\mathcal{M}) = \underline{g}_* \underline{f}^{-1}(\mathcal{M}), \quad (1.2)$$

called the Radon transform.

Lemma 1.2. *Let \mathcal{M} be a \mathcal{D}_X -module.*

(i) *We have*

$$\begin{aligned} R(\mathcal{M}) &= Rg_*((\mathcal{D}_{Y \leftarrow S} \otimes_{\mathcal{D}_S} \mathcal{D}_{S \rightarrow X}) \otimes_{f^{-1}\mathcal{D}_X}^L f^{-1}\mathcal{M}) \\ &= Rg_*(f^{-1}(\Omega_X \otimes_{\mathcal{O}_X} \mathcal{M}) \otimes_{f^{-1}\mathcal{D}_X}^L (\mathcal{D}_{X \times Y \leftarrow S} \otimes_{\mathcal{D}_S} \mathcal{O}_S)). \end{aligned}$$

(ii) *If S is closed in $X \times Y$, then we have*

$$R(\mathcal{M}) = \underline{q2}_*(\underline{q1}^{-1}\mathcal{M} \otimes_{\mathcal{O}_{X \times Y}}^L \mathcal{B}_{S|X \times Y}).$$

Proof. (i) follows from the definition and Lemma 1.1, and (ii) is a consequence of the projection formula for \mathcal{D} -modules. \square

Let us consider the special case where $\mathcal{M} = \mathcal{DL} = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{L}$. By Lemma 1.2 we have the following.

Lemma 1.3. *Let \mathcal{L} be a locally free \mathcal{O}_X -module.*

(i) *We have*

$$\begin{aligned} R(\mathcal{DL}) &= Rg_*((\mathcal{D}_{Y \leftarrow S} \otimes_{\mathcal{D}_S} \mathcal{D}_{S \rightarrow X}) \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{L}) \\ &= Rg_*(f^{-1}(\Omega_X \otimes_{\mathcal{O}_X} \mathcal{L}) \otimes_{f^{-1}\mathcal{O}_X} (\mathcal{D}_{X \times Y \leftarrow S} \otimes_{\mathcal{D}_S} \mathcal{O}_S)). \end{aligned}$$

(ii) *If S is closed in $X \times Y$, then we have*

$$R(\mathcal{DL}) = Rq_{2*}(q_1^{-1}(\Omega_X \otimes_{\mathcal{O}_X} \mathcal{L}) \otimes_{q_1^{-1}\mathcal{O}_X} \mathcal{B}_{S|X \times Y}).$$

An immediate consequence of Lemma 1.3(i) is:

Lemma 1.4. *For any locally free \mathcal{O}_X -module \mathcal{L} we have $H^p(R(\mathcal{DL})) = 0$ for any $p < 0$.*

Definition 1.5. Let \mathcal{L} (resp. \mathcal{L}') be a locally free \mathcal{O}_X - (resp. \mathcal{O}_Y -) module. We say that the pair $(\mathcal{L}, \mathcal{L}')$ is an extremal case for the correspondence (1.1) if there is an \mathcal{O}_S -linear isomorphism

$$\Omega_g \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{L} \simeq g^*\mathcal{L}'.$$

Proposition 1.6. *Let $(\mathcal{L}, \mathcal{L}')$ be an extremal case for (1.1). Then there exists a natural nontrivial \mathcal{D}_Y -linear morphism*

$$\mathcal{DL}' \rightarrow H^0(R(\mathcal{DL})). \quad (1.3)$$

Proof. The canonical morphism $\Omega_g \rightarrow \mathcal{D}_{Y \leftarrow S} \otimes_{\mathcal{D}_S} \mathcal{D}_{S \rightarrow X}$ induces a monomorphism

$$g^*\mathcal{L}' \simeq \Omega_g \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{L} \rightarrow \mathcal{D}_{Y \leftarrow S} \otimes_{\mathcal{D}_S} \mathcal{D}_{S \rightarrow X} \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{L}$$

of $g^{-1}\mathcal{O}_Y$ -modules. Applying g_* we obtain a sequence of morphisms

$$\begin{aligned} \mathcal{L}' &\rightarrow \mathcal{L}' \otimes_{\mathcal{O}_Y} g_*\mathcal{O}_S \simeq g_*(g^*\mathcal{L}') \\ &\rightarrow g_*(\mathcal{D}_{Y \leftarrow S} \otimes_{\mathcal{D}_S} \mathcal{D}_{S \rightarrow X} \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{L}) = H^0(R(\mathcal{DL})) \end{aligned}$$

of \mathcal{O}_Y -modules. The morphism $\mathcal{L}' \rightarrow \mathcal{L}' \otimes_{\mathcal{O}_Y} g_*\mathcal{O}_S$ is nontrivial by the definition, and the morphism $g_*(g^*\mathcal{L}') \rightarrow g_*(\mathcal{D}_{Y \leftarrow S} \otimes_{\mathcal{D}_S} \mathcal{D}_{S \rightarrow X} \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{L})$ is a monomorphism by the left exactness of g_* . Thus the composition $\mathcal{L}' \rightarrow H^0(R(\mathcal{DL}))$ is nontrivial. Hence it induces a canonical nontrivial morphism $\mathcal{DL}' \rightarrow H^0(R(\mathcal{DL}))$ of \mathcal{D}_Y -modules. \square

1.3 Adjunction formulas

In this subsection we consider topological problems, and hence we work in the analytic category rather than the algebraic category.

For a complex manifold Z we denote by \mathcal{O}_Z the sheaf of holomorphic functions on Z and by \mathcal{D}_Z the sheaf of holomorphic differential operators. For an algebraic

manifold Z over \mathbb{C} we denote the corresponding complex manifold by Z_{an} , and for a morphism $f : Z \rightarrow Z'$ of algebraic manifolds we denote the corresponding holomorphic map by $f_{\text{an}} : Z_{\text{an}} \rightarrow Z'_{\text{an}}$. For an algebraic manifold Z and an \mathcal{O}_Z -module \mathcal{F} we set $\mathcal{F}_{\text{an}} = \mathcal{O}_{Z_{\text{an}}} \otimes_{\mathcal{O}_Z} \mathcal{F}$.

In the correspondence (1.1), let us consider also a functor in the derived category $\mathbf{D}^b(\mathbb{C})$ of sheaves of \mathbb{C} -vector spaces, going in the opposite direction:

$$r : \mathbf{D}^b(\mathbb{C}_{Y_{\text{an}}}) \rightarrow \mathbf{D}^b(\mathbb{C}_{X_{\text{an}}}), \quad r(F) = Rg_{\text{an}*}f_{\text{an}}^{-1}(F).$$

For example, let D be a Zariski locally closed subset of Y_{an} and take $F = \mathbb{C}_D$ (the constant sheaf with fiber \mathbb{C} on D and zero on $Y_{\text{an}} \setminus D$): then, for any $x \in X$ one has

$$r(\mathbb{C}_D)_x \simeq \text{R}\Gamma_c(S_{D,x}; \mathbb{C}_{S_{D,x}}), \quad S_{D,x} = \{y \in D : (x, y) \in S\}. \quad (1.4)$$

One has the following “adjunction formulas” (see [5]).

Proposition 1.7. *Let \mathcal{L} be a locally free \mathcal{O}_X -module and let $F \in \mathbf{D}^b(\mathbb{C}_{Y_{\text{an}}})$. Then, setting $l = \dim Y - \dim S$ and $m = \dim S + \dim Y - 2 \dim X$, we have*

$$\text{R}\Gamma(X_{\text{an}}; r(F) \otimes \mathcal{L}_{\text{an}}^*) \simeq \text{RHom}_{\mathcal{D}_{Y_{\text{an}}}}(R(\mathcal{DL})_{\text{an}}, F \otimes \mathcal{O}_{Y_{\text{an}}})[l], \quad (1.5)$$

$$\text{RHom}(r(F), \mathcal{L}_{\text{an}}^*) \simeq \text{RHom}_{\mathcal{D}_{Y_{\text{an}}}}(R(\mathcal{DL})_{\text{an}}, R\mathcal{H}om(F, \mathcal{O}_{Y_{\text{an}}}))[m]. \quad (1.6)$$

Once the calculation of $R(\mathcal{DL})$ has been performed, these formulas will give different applications by computing $r(F)$ for different choices of the sheaf F (a problem of geometric nature).

1.4 Quasi-equivariant \mathcal{D} -modules

Let us recall the definition of (quasi-)equivariant \mathcal{D} -modules (we refer to Kashiwara [7]).

Let G be an algebraic group over \mathbb{C} , and let \mathfrak{g} be its Lie algebra. We denote the enveloping algebra of \mathfrak{g} by $\mathcal{U}(\mathfrak{g})$. Let Z be a G -manifold, i.e. an algebraic manifold endowed with an action of G . Let us denote by $\mu : G \times Z \rightarrow Z$ the action $\mu(g, z) = gz$ and by $p : G \times Z \rightarrow Z$ the projection $p(g, z) = z$. Moreover, define the morphisms $q_j : G \times G \times Z \rightarrow G \times Z$ ($j = 1, 2, 3$) by $q_1(g_1, g_2, z) = (g_1, g_2 z)$, $q_2(g_1, g_2, z) = (g_1 g_2, z)$ and $q_3(g_1, g_2, z) = (g_2, z)$, and observe that $\mu \circ q_1 = \mu \circ q_2$, $p \circ q_2 = p \circ q_3$ and $\mu \circ q_3 = p \circ q_1$.

A G -equivariant \mathcal{O}_Z -module is an \mathcal{O}_Z -module \mathcal{M} endowed with an $\mathcal{O}_{G \times Z}$ -linear isomorphism $\beta : \mu^* \mathcal{M} \rightarrow p^* \mathcal{M}$ such that the following diagram commutes:

$$\begin{array}{ccc} q_2^* \mu^* \mathcal{M} & \xrightarrow{q_2^* \beta} & q_2^* p^* \mathcal{M} \\ \wr \downarrow & & \wr \downarrow \\ q_1^* \mu^* \mathcal{M} & \xrightarrow{q_1^* \beta} q_1^* p^* \mathcal{M} \cong q_3^* \mu^* \mathcal{M} \xrightarrow{q_3^* \beta} & q_3^* p^* \mathcal{M}. \end{array}$$

For a G -equivariant \mathcal{O}_Z -module \mathcal{M} we have a canonical Lie algebra homomorphism $\rho_{\mathcal{M}} : \mathfrak{g} \rightarrow \text{End}_{\mathbb{C}}(\mathcal{M})$.

Let $\mathcal{O}_G \boxtimes \mathcal{D}_Z$ denote the subalgebra $\mathcal{O}_{G \times Z} \otimes_{p^{-1}\mathcal{O}_Z} p^{-1}\mathcal{D}_Z$ of $\mathcal{D}_{G \times Z}$. A \mathcal{D}_Z -module \mathcal{M} is called G -equivariant (resp. quasi- G -equivariant) if it is endowed with a G -equivariant \mathcal{O}_Z -module structure such that the isomorphism $\beta : \mu^*\mathcal{M} \rightarrow p^*\mathcal{M}$ is $\mathcal{D}_{G \times Z}$ -linear (resp. $\mathcal{O}_G \boxtimes \mathcal{D}_Z$ -linear). Note that for a morphism $f : Z \rightarrow Z'$ of algebraic manifolds and a $\mathcal{D}_{Z'}$ -module \mathcal{M} the \mathcal{D}_Z -module $H^0(\underline{f}^{-1}\mathcal{M})$ is naturally isomorphic to $f^*\mathcal{M}$ as an \mathcal{O}_Z -module.

For example for a G -equivariant \mathcal{O}_Z -module \mathcal{F} the \mathcal{D}_Z -module $\mathcal{D}_Z \otimes_{\mathcal{O}_Z} \mathcal{F}$ is endowed with a natural quasi- G -equivariant \mathcal{D}_Z -module structure.

We denote by $\text{Mod}_G(\mathcal{D}_Z)$ the category of quasi- G -equivariant \mathcal{D}_Z -modules, and by $\mathbf{D}_G^b(\mathcal{D}_Z)$ the derived category of \mathcal{D}_Z -modules with bounded quasi- G -equivariant cohomology (see Kashiwara-Schmid [8]).

Let \mathcal{M} be a quasi- G -equivariant \mathcal{D}_Z -module. The canonical Lie algebra homomorphism $\mathfrak{g} \rightarrow \mathcal{D}_Z$ induces a Lie algebra homomorphism $\kappa_{\mathcal{M}} : \mathfrak{g} \rightarrow \text{End}_{\mathbb{C}}(\mathcal{M})$. Set $\gamma_{\mathcal{M}} = \rho_{\mathcal{M}} - \kappa_{\mathcal{M}}$.

Proposition 1.8 (Kashiwara [7]). (i) *We have $\gamma_{\mathcal{M}}(a) \in \text{End}_{\mathcal{D}_Z}(\mathcal{M})$ for any $a \in \mathfrak{g}$.*

(ii) *The linear map $\gamma_{\mathcal{M}} : \mathfrak{g} \rightarrow \text{End}_{\mathcal{D}_Z}(\mathcal{M})$ is a Lie algebra homomorphism.*

(iii) *We have $\gamma_{\mathcal{M}} = 0$ if and only if \mathcal{M} is G -equivariant.*

We also denote by

$$\gamma_{\mathcal{M}} : \mathcal{U}(\mathfrak{g}) \rightarrow \text{End}_{\mathcal{D}_Z}(\mathcal{M}) \quad (1.7)$$

the corresponding algebra homomorphism.

Fix $x \in Z$ and set $H = \{g \in G : gx = x\}$. For a G -equivariant \mathcal{O}_Z -module \mathcal{M} , the fiber

$$\mathcal{M}(x) = \mathbb{C} \otimes_{\mathcal{O}_{Z,x}} \mathcal{M}_x$$

of \mathcal{M} at x is endowed with a natural H -module structure. If \mathcal{M} is a quasi- G -equivariant \mathcal{D}_Z -module, then $\mathcal{M}(x)$ is also endowed with a \mathfrak{g} -module structure induced from the \mathcal{O}_Z -linear action $\gamma_{\mathcal{M}}$. For $M = \mathcal{M}(x)$ we have the following.

- (a) the action of the Lie algebra of H on M given by differentiating the H -module structure coincides with the restriction of the action of \mathfrak{g} ,
- (b) $hum = (\text{Ad}(h)u)hm$ for any $h \in H$, $u \in \mathfrak{g}$, $m \in M$.

Here Ad denotes the adjoint action. A vector space M equipped with structures of an H -modules and a \mathfrak{g} -module is called a (\mathfrak{g}, H) -module if it satisfies the conditions (a) and (b) above.

The following result plays a crucial role in the rest of this paper.

Proposition 1.9. *Assume that $Z = G/H$, where H is a closed subgroup of G , and set $x = eH \in Z$.*

- (i) *The category of G -equivariant \mathcal{O}_Z -modules is equivalent to the category of H -modules via the correspondence $\mathcal{M} \mapsto \mathcal{M}(x)$.*
- (ii) *The category of quasi- G -equivariant \mathcal{D}_Z -modules is equivalent to the category of (\mathfrak{g}, H) -modules via the correspondence $\mathcal{M} \mapsto \mathcal{M}(x)$.*

The statement (i) is well-known (see [11]), and (ii) is due to Kashiwara [7].

2 Radon transforms for generalized flag manifolds

2.1 Quasi-equivariant \mathcal{D} -modules on generalized flag manifolds

Let G be a connected reductive algebraic group over \mathbb{C} , and \mathfrak{g} the Lie algebra of G . The group G acts on \mathfrak{g} by the adjoint action Ad . Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} , Δ the root system in \mathfrak{h}^* , $\{\alpha_i : i \in I_0\}$ a set of simple roots, Δ^+ the set of positive roots, Δ^- the set of negative roots, $\mathfrak{h}_{\mathbb{Z}}^* = \text{Hom}(H, \mathbb{C}^\times) \subset \mathfrak{h}^*$ the weight lattice, and W the Weyl group. For $\alpha \in \Delta$ we denote by \mathfrak{g}_α the corresponding root space and by $\alpha^\vee \in \mathfrak{h}$ the corresponding coroot. For $i \in I_0$ we denote by $s_i \in W$ the reflection corresponding to i . For $w \in W$ we set $\ell(w) = \sharp(w\Delta^- \cap \Delta^+)$. Set $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$, and define a (shifted) affine action of W on \mathfrak{h}^* by

$$w \circ \lambda = w(\lambda + \rho) - \rho. \quad (2.1)$$

For $I \subset I_0$, we set

$$\begin{aligned} \Delta_I &= \Delta \cap \sum_{i \in I} \mathbb{Z}\alpha_i, & \Delta_I^+ &= \Delta_I \cap \Delta^+, & W_I &= \langle s_i : i \in I \rangle \subset W \\ \mathfrak{l}_I &= \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_I} \mathfrak{g}_\alpha \right), & \mathfrak{n}_I &= \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_\alpha, & \mathfrak{p}_I &= \mathfrak{l}_I \oplus \mathfrak{n}_I, \\ (\mathfrak{h}_{\mathbb{Z}}^*)_I &= \{ \lambda \in \mathfrak{h}_{\mathbb{Z}}^* : \lambda(\alpha_i^\vee) \geq 0 \text{ for any } i \in I \}, \\ (\mathfrak{h}_{\mathbb{Z}}^*)_I^0 &= \{ \lambda \in \mathfrak{h}_{\mathbb{Z}}^* : \lambda(\alpha_i^\vee) = 0 \text{ for any } i \in I \} \subset (\mathfrak{h}_{\mathbb{Z}}^*)_I, \\ \rho_I &= \left(\sum_{\alpha \in \Delta^+ \setminus \Delta_I} \alpha \right) / 2. \end{aligned}$$

We denote by w_I the longest element of W_I . It is an element of W_I characterized by $w_I(\Delta_I^-) = \Delta_I^+$. Let L_I , N_I and P_I be the subgroups of G corresponding to \mathfrak{l}_I , \mathfrak{n}_I and \mathfrak{p}_I .

For $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I$ let $V_I(\lambda)$ be the irreducible L_I -module with highest weight λ . We regard $V_I(\lambda)$ as a P_I -module with the trivial action of N_I , and define the generalized Verma module with highest weight λ by

$$M_I(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}_I)} V_I(\lambda). \quad (2.2)$$

Let $L(\lambda)$ be the unique irreducible quotient of $M_I(\lambda)$ (note that $L(\lambda)$ does not depend on the choice of I such that $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I$). Then any irreducible P_I -module is isomorphic to $V_I(\lambda)$ for some $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I$, and we have $\dim V_I(\lambda) = 1$ if and only if $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I^0$. Moreover, any irreducible (\mathfrak{g}, P_I) -module is isomorphic to $L(\lambda)$ for some $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I$.

Let

$$X_I = G/P_I$$

be the generalized flag manifold associated to I .

By the category equivalence given in Proposition 1.9 isomorphism classes of G -equivariant \mathcal{O}_{X_I} -modules (resp. quasi- G -equivariant \mathcal{D}_{X_I} -modules) are in one-to-one correspondence with isomorphism classes of P_I -modules (resp. (\mathfrak{g}, P_I) -modules). For $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I$ we denote by $\mathcal{O}_{X_I}(\lambda)$ the G -equivariant \mathcal{O}_{X_I} -module corresponding to the irreducible P_I -module $V_I(\lambda)$. We see easily the following.

Lemma 2.1. *Let $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I$. The quasi- G -equivariant \mathcal{D}_{X_I} -module corresponding to the (\mathfrak{g}, P_I) -module $M_I(\lambda)$ is isomorphic to $\mathcal{D}\mathcal{O}_{X_I}(\lambda) = \mathcal{D}_{X_I} \otimes_{\mathcal{O}_{X_I}} \mathcal{O}_{X_I}(\lambda)$.*

We need the following relative version of the Borel-Weil-Bott theorem later (see Bott [3]).

Proposition 2.2. *Let $I \subset J \subset I_0$ and let $\pi : X_I \rightarrow X_J$ be the canonical projection. For $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I$ we have the following.*

- (i) *If there exists some $\alpha \in \Delta_J$ satisfying $(\lambda + \rho - 2\rho_I)(\alpha^\vee) = 0$, then we have $R\pi_*(\mathcal{O}_{X_I}(\lambda)) = 0$.*
- (ii) *Assume that $(\lambda + \rho - 2\rho_I)(\alpha^\vee) \neq 0$ for any $\alpha \in \Delta_J$. Take $w \in W_J$ satisfying $(w(\lambda + \rho - 2\rho_I))(\alpha^\vee) > 0$ for any $\alpha \in \Delta_J^+$. Then we have*

$$R\pi_*(\mathcal{O}_{X_I}(\lambda)) = \mathcal{O}_{X_J}(w(\lambda + \rho - 2\rho_I) - (\rho - 2\rho_J))[-(\ell(w_J w) - \ell(w_I))].$$

Let $I, J \subset I_0$ with $I \neq J$. The diagonal action of G on $X_I \times X_J$ has a finite number of orbits, and the only closed one $G(eP_I, eP_J)$ is identified with $X_{I \cap J} = G/(P_I \cap P_J)$. In the rest of this paper we shall consider the correspondence (1.1) for $X = X_I$, $Y = X_J$ and $S = X_{I \cap J}$:

$$X_I \xleftarrow{f} X_{I \cap J} \xrightarrow{g} X_J \quad (2.3)$$

and the Radon transform $R(\mathcal{D}\mathcal{O}_{X_I}(\lambda))$ for $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I$. Since f and g are morphisms of G -manifolds, the functor (1.2) induces a functor

$$R : \mathbf{D}_G^b(\mathcal{D}_{X_I}) \rightarrow \mathbf{D}_G^b(\mathcal{D}_{X_J}). \quad (2.4)$$

Note that we have

$$\Omega_g \simeq \mathcal{O}_{X_{I \cap J}}(\gamma_{I,J}) \quad \text{for } \gamma_{I,J} = \sum_{\alpha \in \Delta_J^+ \setminus \Delta_I} \alpha. \quad (2.5)$$

2.2 Radon transforms of quasi-equivariant \mathcal{D} -modules

Let $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I$. We describe our method to analyze $R(\mathcal{DO}_{X_I}(\lambda)) = \underline{g}_* \underline{f}^{-1}(\mathcal{DO}_{X_I}(\lambda))$.

By

$$(\underline{f}^{-1}(\mathcal{DO}_{X_I}(\lambda)))(e(P_I \cap P_J)) \simeq \mathcal{DO}_{X_I}(\lambda)(eP_I) \simeq M_I(\lambda)$$

the quasi- G -equivariant $\mathcal{D}_{X_{I \cap J}}$ -module $\underline{f}^{-1}(\mathcal{DO}_{X_I}(\lambda))$ corresponds to the $(\mathfrak{g}, P_I \cap P_J)$ -module $M_I(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}_I)} V_I(\lambda)$ under the category equivalence given in Proposition 1.9.

Set

$$\Gamma = \{x \in W_I : x \text{ is the shortest element of } W_{I \cap J}x\}, \quad (2.6)$$

$$\Gamma_k = \{x \in \Gamma : \ell(x) = k\}. \quad (2.7)$$

It is well-known that an element $x \in W_I$ belongs to Γ if and only if $x^{-1}\Delta_{I \cap J}^+ \subset \Delta_I^+$. This condition is also equivalent to

$$(x(\lambda + \rho))(\alpha^\vee) > 0 \text{ for any } \alpha \in \Delta_{I \cap J}^+. \quad (2.8)$$

In particular, we have $x \circ \lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_{I \cap J}$ for $x \in \Gamma$.

By Lepowsky [9] and Rocha-Caridi [13] we have the following resolution of the finite dimensional \mathfrak{l}_I -module $V_I(\lambda)$:

$$0 \rightarrow N_n \rightarrow N_{n-1} \rightarrow \cdots \rightarrow N_1 \rightarrow N_0 \rightarrow V_I(\lambda) \rightarrow 0 \quad (2.9)$$

with $n = \dim \mathfrak{l}_I / \mathfrak{l}_I \cap \mathfrak{p}_J$ and

$$N_k = \bigoplus_{x \in \Gamma_k} \mathcal{U}(\mathfrak{l}_I) \otimes_{\mathcal{U}(\mathfrak{l}_I \cap \mathfrak{p}_J)} V_{I \cap J}(x \circ \lambda).$$

By the Poincaré-Birkhoff-Witt theorem we have the isomorphism

$$\mathcal{U}(\mathfrak{l}_I) \otimes_{\mathcal{U}(\mathfrak{l}_I \cap \mathfrak{p}_J)} V_{I \cap J}(x \circ \lambda) \simeq \mathcal{U}(\mathfrak{p}_I) \otimes_{\mathcal{U}(\mathfrak{p}_{I \cap J})} V_{I \cap J}(x \circ \lambda)$$

of $\mathcal{U}(\mathfrak{l}_I)$ -modules, where $\mathfrak{n}_{I \cap J}$ acts trivially on $V_{I \cap J}(x \circ \lambda)$. Moreover, the action of \mathfrak{n}_I on $\mathcal{U}(\mathfrak{p}_I) \otimes_{\mathcal{U}(\mathfrak{p}_{I \cap J})} V_{I \cap J}(x \circ \lambda)$ is trivial. Indeed, by $[\mathfrak{p}_I, \mathfrak{n}_I] \subset \mathfrak{n}_I$ we have $\mathfrak{n}_I \mathcal{U}(\mathfrak{p}_I) = \mathcal{U}(\mathfrak{p}_I) \mathfrak{n}_I$, and hence

$$\mathfrak{n}_I(\mathcal{U}(\mathfrak{p}_I) \otimes_{\mathcal{U}(\mathfrak{p}_{I \cap J})} V_{I \cap J}(x \circ \lambda)) \subset \mathcal{U}(\mathfrak{p}_I) \mathfrak{n}_I \otimes V_{I \cap J}(x \circ \lambda) \subset \mathcal{U}(\mathfrak{p}_I) \otimes \mathfrak{n}_I V_{I \cap J}(x \circ \lambda) = 0$$

by $\mathfrak{n}_I \subset \mathfrak{n}_{I \cap J}$. Thus we obtain the following resolution of the finite dimensional \mathfrak{p}_I -module $V_I(\lambda)$ (with trivial action of \mathfrak{n}_I):

$$0 \rightarrow N'_n \rightarrow N'_{n-1} \rightarrow \cdots \rightarrow N'_1 \rightarrow N'_0 \rightarrow V_I(\lambda) \rightarrow 0 \quad (2.10)$$

with

$$N'_k = \bigoplus_{x \in \Gamma_k} \mathcal{U}(\mathfrak{p}_I) \otimes_{\mathcal{U}(\mathfrak{p}_{I \cap J})} V_{I \cap J}(x \circ \lambda).$$

By tensoring $\mathcal{U}(\mathfrak{g})$ to (2.10) over $\mathcal{U}(\mathfrak{p}_I)$ we obtain the following resolution of the $(\mathfrak{g}, P_{I \cap J})$ -module $M_I(\lambda)$:

$$0 \rightarrow \tilde{N}_n \rightarrow \tilde{N}_{n-1} \rightarrow \cdots \rightarrow \tilde{N}_1 \rightarrow \tilde{N}_0 \rightarrow M_I(\lambda) \rightarrow 0 \quad (2.11)$$

with

$$\tilde{N}_k = \bigoplus_{x \in \Gamma_k} M_{I \cap J}(x \circ \lambda).$$

Since the quasi- G -equivariant $\mathcal{D}_{X_{I \cap J}}$ -module corresponding to the $(\mathfrak{g}, P_{I \cap J})$ -module $M_{I \cap J}(x \circ \lambda)$ is $\mathcal{DO}_{X_{I \cap J}}(x \circ \lambda)$, we have obtained the following resolution of the quasi- G -equivariant $\mathcal{D}_{X_{I \cap J}}$ -module $\underline{f}^{-1}(\mathcal{DO}_{X_I}(\lambda))$:

$$0 \rightarrow \mathcal{N}_n \rightarrow \mathcal{N}_{n-1} \rightarrow \cdots \rightarrow \mathcal{N}_1 \rightarrow \mathcal{N}_0 \rightarrow \underline{f}^{-1}(\mathcal{DO}_{X_I}(\lambda)) \rightarrow 0 \quad (2.12)$$

with

$$\mathcal{N}_k = \bigoplus_{x \in \Gamma_k} \mathcal{DO}_{X_{I \cap J}}(x \circ \lambda). \quad (2.13)$$

Our next task is to investigate on $\underline{g}_*(\mathcal{DO}_{X_{I \cap J}}(x \circ \lambda))$ for $x \in \Gamma$. We first remark that

$$\underline{g}_*(\mathcal{DO}_{X_{I \cap J}}(x \circ \lambda)) = \mathcal{D}_{X_J} \otimes_{\mathcal{O}_{X_J}} Rg_*(\mathcal{O}_{X_{I \cap J}}(x \circ \lambda + \gamma_{I,J})). \quad (2.14)$$

Indeed, by (2.5) we have

$$\begin{aligned} \underline{g}_*(\mathcal{DO}_{X_{I \cap J}}(x \circ \lambda)) &= Rg_*(\mathcal{D}_{X_J \leftarrow X_{I \cap J}} \otimes_{\mathcal{D}_{X_{I \cap J}}}^L \mathcal{D}_{X_{I \cap J}} \otimes_{\mathcal{O}_{X_{I \cap J}}}^L \mathcal{O}_{X_{I \cap J}}(x \circ \lambda)) \\ &= Rg_*(\mathcal{D}_{X_J \leftarrow X_{I \cap J}} \otimes_{\mathcal{O}_{X_{I \cap J}}}^L \mathcal{O}_{X_{I \cap J}}(x \circ \lambda)) \\ &= Rg_*(g^{-1} \mathcal{D}_{X_J} \otimes_{g^{-1} \mathcal{O}_{X_J}} \Omega_g \otimes_{\mathcal{O}_{X_{I \cap J}}} \mathcal{O}_{X_{I \cap J}}(x \circ \lambda)) \\ &= \mathcal{D}_{X_J} \otimes_{\mathcal{O}_{X_J}} Rg_*(\Omega_g \otimes_{\mathcal{O}_{X_{I \cap J}}} \mathcal{O}_{X_{I \cap J}}(x \circ \lambda)) \\ &= \mathcal{D}_{X_J} \otimes_{\mathcal{O}_{X_J}} Rg_*(\mathcal{O}_{X_{I \cap J}}(x \circ \lambda + \gamma_{I,J})). \end{aligned}$$

Lemma 2.3. *Let $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I$ and $x \in \Gamma$.*

- (i) *If $(x(\lambda + \rho))(\alpha^\vee) = 0$ for some $\alpha \in \Delta_J$, then we have $Rg_*(\mathcal{O}_{X_{I \cap J}}(x \circ \lambda + \gamma_{I,J})) = 0$.*
- (ii) *Assume that $(x(\lambda + \rho))(\alpha^\vee) \neq 0$ for any $\alpha \in \Delta_J$. Take $y \in W_J$ satisfying $(yx(\lambda + \rho))(\alpha^\vee) > 0$ for any $\alpha \in \Delta_J^+$. Then we have*

$$Rg_*(\mathcal{O}_{X_{I \cap J}}(x \circ \lambda + \gamma_{I,J})) = \mathcal{O}_{X_J}((yx) \circ \lambda) [-(\ell(w_J y) - \ell(w_{I \cap J}))].$$

Proof. Since $\Delta^+ \setminus \Delta_J$ is stable under the action of W_J , we have $y\rho_J = \rho_J$ for any $y \in W_J$. In particular,

$$\rho_J = s_\alpha(\rho_J) = \rho_J - \rho_J(\alpha^\vee)\alpha$$

for any $\alpha \in \Delta_J$, and hence $\rho_J(\alpha^\vee) = 0$ for any $\alpha \in \Delta_J$.

By the definition we have

$$x \circ \lambda + \gamma_{I,J} + \rho - 2\rho_{I \cap J} = x(\lambda + \rho) + \gamma_{I,J} - 2\rho_{I \cap J} = x(\lambda + \rho) - 2\rho_J,$$

and

$$y(x(\lambda + \rho) - 2\rho_J) - (\rho - 2\rho_J) = yx(\lambda + \rho) - 2\rho_J - (\rho - 2\rho_J) = (yx) \circ \lambda$$

for any $y \in W_J$. Hence the assertion follows from Proposition 2.2. \square

Set

$$\Gamma(\lambda) = \{x \in \Gamma : (x(\lambda + \rho))(\alpha^\vee) \neq 0 \text{ for any } \alpha \in \Delta_J\}, \quad (2.15)$$

$$\Gamma_k(\lambda) = \{x \in \Gamma(\lambda) : \ell(x) = k\}. \quad (2.16)$$

and for $x \in \Gamma(\lambda)$ denote by y_x the element of W_J satisfying $(y_x x(\lambda + \rho))(\alpha^\vee) > 0$ for any $\alpha \in \Delta_J^+$. Set

$$m(x) = \ell(w_J y_x) - \ell(w_{I \cap J}) \quad \text{for } x \in \Gamma(\lambda). \quad (2.17)$$

Lemma 2.4. *For $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I$ and $x \in \Gamma(\lambda)$ we have*

$$\ell(x) = \sharp\{\alpha \in \Delta_J^+ \setminus \Delta_J : (x(\lambda + \rho))(\alpha^\vee) < 0\}, \quad (2.18)$$

$$m(x) = \sharp\{\alpha \in \Delta_J^+ \setminus \Delta_I : (x(\lambda + \rho))(\alpha^\vee) > 0\}. \quad (2.19)$$

Proof. We have

$$\begin{aligned} \ell(x) &= \sharp(x^{-1}\Delta_I^- \cap \Delta_I^+) \\ &= \sharp\{\alpha \in \Delta_I^+ : (x(\lambda + \rho))(\alpha^\vee) < 0\} \\ &= \sharp\{\alpha \in \Delta_J^+ \setminus \Delta_J : (x(\lambda + \rho))(\alpha^\vee) < 0\}, \end{aligned}$$

and

$$\begin{aligned} m(x) &= \ell(w_J) - \ell(y_x) - \ell(w_{I \cap J}) \\ &= \sharp(\Delta_J^+ \setminus \Delta_I) - \sharp(y_x^{-1}\Delta_J^- \cap \Delta_J^+) \\ &= \sharp(\Delta_J^+ \setminus \Delta_I) - \sharp\{\alpha \in \Delta_J^+ : (x(\lambda + \rho))(\alpha^\vee) < 0\} \\ &= \sharp(\Delta_J^+ \setminus \Delta_I) - \sharp\{\alpha \in \Delta_J^+ \setminus \Delta_I : (x(\lambda + \rho))(\alpha^\vee) < 0\} \\ &= \sharp\{\alpha \in \Delta_J^+ \setminus \Delta_I : (x(\lambda + \rho))(\alpha^\vee) > 0\} \end{aligned}$$

by (2.8). \square

Proposition 2.5. *Let $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I$. Then there exists a family $\{\mathcal{M}(k)^{\bullet}\}_{k \geq 0}$ of objects of $\mathbf{D}_G^b(\mathcal{D}_{X_J})$ satisfying the following conditions.*

- (i) $\mathcal{M}(0)^{\bullet} \simeq R(\mathcal{D}\mathcal{O}_{X_I}(\lambda))$.
- (ii) $\mathcal{M}(k)^{\bullet} = 0$ for $k > \dim \mathfrak{l}_I / \mathfrak{l}_I \cap \mathfrak{p}_J$.
- (iii) *We have a distinguished triangle*

$$\mathcal{C}(k)^{\bullet} \rightarrow \mathcal{M}(k)^{\bullet} \rightarrow \mathcal{M}(k+1)^{\bullet} \xrightarrow{+1}$$

where

$$\mathcal{C}(k)^{\bullet} = \bigoplus_{x \in \Gamma_k(\lambda)} \mathcal{D}\mathcal{O}_{X_J}((y_x x) \circ \lambda)[\ell(x) - m(x)].$$

Proof. For $0 \leq k \leq \dim \mathfrak{l}_I / \mathfrak{l}_I \cap \mathfrak{p}_J$ define an object $\mathcal{N}(k)^{\bullet}$ of $\mathbf{D}_G^b(\mathcal{D}_{X_{I \cap J}})$ by

$$\mathcal{N}(k)^{\bullet} = [\cdots \rightarrow 0 \rightarrow \mathcal{N}_n \rightarrow \mathcal{N}_{n-1} \rightarrow \cdots \rightarrow \mathcal{N}_k \rightarrow 0 \cdots],$$

where \mathcal{N}_j has degree $-j$ (see (2.12) and (2.13) for the notation). For $k > \dim \mathfrak{l}_I / \mathfrak{l}_I \cap \mathfrak{p}_J$ we set $\mathcal{N}(k)^{\bullet} = 0$. By $\mathcal{N}(0)^{\bullet} \simeq \underline{f}^{-1}(\mathcal{D}\mathcal{O}_{X_I}(\lambda))$ we have $\underline{g}_* \mathcal{N}(0)^{\bullet} \simeq R(\mathcal{D}\mathcal{O}_{X_I}(\lambda))$. Set $\mathcal{M}(k)^{\bullet} = \underline{g}_* \mathcal{N}(k)^{\bullet}$. Then the statements (i) and (ii) are obvious. Let us show (iii). Applying \underline{g}_* to the distinguished triangle

$$\mathcal{N}_k[k] \rightarrow \mathcal{N}(k)^{\bullet} \rightarrow \mathcal{N}(k+1)^{\bullet} \xrightarrow{+1}$$

we obtain a distinguished triangle

$$\underline{g}_* \mathcal{N}_k[k] \rightarrow \mathcal{M}(k)^{\bullet} \rightarrow \mathcal{M}(k+1)^{\bullet} \xrightarrow{+1}.$$

By (2.13), (2.14) and Lemma 2.3 we have

$$\underline{g}_* \mathcal{N}_k = \bigoplus_{x \in \Gamma_k(\lambda)} \mathcal{D}\mathcal{O}_{X_J}((y_x x) \circ \lambda)[-m(x)].$$

The statement (iii) is proved. □

Theorem 2.6. *Let $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I$.*

- (i) *We have*

$$\sum_p (-1)^p [H^p(R(\mathcal{D}\mathcal{O}_{X_I}(\lambda)))] = \sum_{x \in \Gamma(\lambda)} (-1)^{\ell(x) - m(x)} [\mathcal{D}\mathcal{O}_{X_J}((y_x x) \circ \lambda)]$$

in the Grothendieck group of the category of quasi- G -equivariant \mathcal{D}_{X_J} -modules.

(ii) If $\Gamma(\lambda) = \emptyset$, then $R(\mathcal{DO}_{X_I}(\lambda)) = 0$.

(iii) If $\Gamma(\lambda)$ consists of a single element x , then

$$R(\mathcal{DO}_{X_I}(\lambda)) = \mathcal{DO}_{X_J}((y_x x) \circ \lambda)[\ell(x) - m(x)].$$

(iv) If $\ell(x) \geq m(x)$ for any $x \in \Gamma(\lambda)$, then we have $H^p(R(\mathcal{DO}_{X_I}(\lambda))) = 0$ unless $p = 0$.

(v) If $(\lambda + \rho)(\alpha^\vee) < 0$ for any $\alpha \in \Delta_J^+ \setminus \Delta_I$, then there exists a canonical morphism

$$\Phi : \mathcal{DO}_{X_J}((w_J w_{I \cap J}) \circ \lambda) \rightarrow H^0(R(\mathcal{DO}_{X_I}(\lambda))).$$

Moreover, Φ is an epimorphism if $\ell(x) > m(x)$ for any $x \in \Gamma(\lambda) \setminus \{e\}$.

Proof. The statements (i), (ii), (iii) are obvious from Proposition 2.5. The statement (iv) follows from Proposition 2.5 and Lemma 1.4. Assume that λ satisfies the assumption in (v). Then we have $e \in \Gamma(\lambda)$ and $y_e = w_J w_{I \cap J}$. Hence (v) follows from Proposition 2.5. \square

Lemma 2.7. (i) The map $W_J \times \Gamma \rightarrow W_J W_I ((y, x) \mapsto yx)$ is bijective.

(ii) For $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I$ we have

$$\{y_x x : x \in \Gamma(\lambda)\} = \{w \in W_J W_I : (w(\lambda + \rho))(\alpha^\vee) > 0 \text{ for any } \alpha \in \Delta_J^+\}$$

and we have

$$\ell(x) - m(x) = \ell(y_x) + \ell(x) - \sharp(\Delta_J^+ \setminus \Delta_I) = \ell(y_x x) - \sharp(\Delta_J^+ \setminus \Delta_I).$$

Proof. (i) is a consequence of the definition of Γ , and the first statement in (ii) follows from (i) and the definition of y_x . By

$$\ell(x) - m(x) = \ell(x) - (\ell(w_J) - \ell(y_x) - \ell(w_{I \cap J})) = \ell(x) + \ell(y_x) - \sharp(\Delta_J^+ \setminus \Delta_I)$$

we have only to show $\ell(y_x x) = \ell(x) + \ell(y_x)$ for $x \in \Gamma(\lambda)$. We have

$$x\Delta^+ \cap \Delta^- = x\Delta_I^+ \cap \Delta_I^- \subset \Delta_I^- \setminus \Delta_{I \cap J} \subset \Delta^- \setminus \Delta_J$$

by $x \in W_I$ and $x^{-1}\Delta_{I \cap J}^+ \subset \Delta_I^+$. Since $w \in W_J$, we obtain $y_x(x\Delta^+ \cap \Delta^-) \subset \Delta^-$. Hence

$$\begin{aligned} \ell(y_x x) &= \sharp(y_x x \Delta^- \cap \Delta^+) \\ &= \sharp(y_x(x\Delta^- \cap \Delta^+) \cap \Delta^+) + \sharp(y_x(x\Delta^- \cap \Delta^-) \cap \Delta^+) \\ &= \sharp(y_x(x\Delta^- \cap \Delta^+) \cap \Delta^+) + \sharp(y_x \Delta^- \cap \Delta^+) - \sharp(y_x(x\Delta^+ \cap \Delta^-) \cap \Delta^+) \\ &= \ell(x) + \ell(y_x). \end{aligned}$$

\square

For $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I$ we set

$$\Xi(\lambda) = \{w \in W_J W_I : (w(\lambda + \rho))(\alpha^\vee) > 0 \text{ for any } \alpha \in \Delta_J^+\}. \quad (2.20)$$

Using Lemma 2.7 above we can reformulate Theorem 2.6 as follows.

Theorem 2.8. *Let $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I$.*

(i) *We have*

$$\sum_p (-1)^p [H^p(R(\mathcal{DO}_{X_I}(\lambda)))] = (-1)^{\sharp(\Delta_J^+ \setminus \Delta_I)} \sum_{w \in \Xi(\lambda)} (-1)^{\ell(w)} [\mathcal{DO}_{X_J}(w \circ \lambda)]$$

in the Grothendieck group of the category of quasi- G -equivariant \mathcal{D}_{X_J} -modules.

(ii) *If $\Xi(\lambda) = \emptyset$, then $R(\mathcal{DO}_{X_I}(\lambda)) = 0$.*

(iii) *If $\Xi(\lambda)$ consists of a single element w , then*

$$R(\mathcal{DO}_{X_I}(\lambda)) = \mathcal{DO}_{X_J}(w \circ \lambda)[\ell(w) - \sharp(\Delta_J^+ \setminus \Delta_I)].$$

(iv) *If $\ell(w) \geq \sharp(\Delta_J^+ \setminus \Delta_I)$ for any $w \in \Xi(\lambda)$, then we have $H^p(R(\mathcal{DO}_{X_I}(\lambda))) = 0$ unless $p = 0$.*

(v) *If $(\lambda + \rho)(\alpha^\vee) < 0$ for any $\alpha \in \Delta_J^+ \setminus \Delta_I$, then there exists a canonical morphism*

$$\Phi : \mathcal{DO}_{X_J}((w_J w_{I \cap J}) \circ \lambda) \rightarrow H^0(R(\mathcal{DO}_{X_I}(\lambda))).$$

Moreover, Φ is an epimorphism if $\ell(w) > \sharp(\Delta_J^+ \setminus \Delta_I)$ for any $w \in \Xi(\lambda) \setminus \{w_J w_{I \cap J}\}$.

Remark 2.9. The following result which is a little weaker than Theorem 2.8(ii) can be obtained by observing that an integral transform for \mathcal{D} -modules with equivariant kernel preserves the infinitesimal character of a quasi-equivariant \mathcal{D} -module (see e.g. [8]):

$$\text{If } (W \circ \lambda) \cap (\mathfrak{h}_{\mathbb{Z}}^*)_J = \emptyset, \text{ then } R(\mathcal{DO}_{X_I}(\lambda)) = 0. \quad (2.21)$$

An advantage of the argument using the infinitesimal character is that it also works for a broader class of integral transforms in equivariant contexts.

Let us briefly recall this argument (suggested to us by M. Kashiwara). Let Z be a G -manifold, denote by $\mathfrak{z}(\mathfrak{g})$ the center of $\mathcal{U}(\mathfrak{g})$ and set $\mathfrak{n}^+ = \mathfrak{n}_\emptyset = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$. One says that a quasi- G -equivariant \mathcal{D}_Z -module \mathcal{M} has infinitesimal character χ (for some $\chi \in \text{Hom}(\mathfrak{z}(\mathfrak{g}), \mathbb{C})$) if $\gamma_{\mathcal{M}}(a)$ is the multiplication by $\chi(a)$ for any $a \in \mathfrak{z}(\mathfrak{g})$. Define a linear map $\sigma : \mathfrak{z}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{h}) \simeq S(\mathfrak{h})$ as the composition of the embedding $\mathfrak{z}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g})$ and the projection $\mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{h})$ with respect to

the direct sum decomposition $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h}) \oplus (\mathfrak{n}^-\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{n}^+)$. Then σ is an injective homomorphism of \mathbb{C} -algebras. For $\lambda \in \mathfrak{h}^*$ define an algebra homomorphism $\chi_\lambda : \mathfrak{z}(\mathfrak{g}) \rightarrow \mathbb{C}$ by $\chi_\lambda(a) = \langle \sigma(a), \lambda \rangle$. By a result of Harish-Chandra, any algebra homomorphism from $\mathfrak{z}(\mathfrak{g})$ to \mathbb{C} coincides with χ_λ for some $\lambda \in \mathfrak{h}^*$, and for $\lambda, \mu \in \mathfrak{h}^*$ one has $\chi_\lambda = \chi_\mu$ if and only if $\mu \in W \circ \lambda$. By the category equivalence of Proposition 1.9, the infinitesimal characters of quasi- G -equivariant \mathcal{D}_{X_I} -modules are of the form χ_λ for $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I$. Therefore, recalling Harish-Chandra's result, if $(W \circ \lambda) \cap (\mathfrak{h}_{\mathbb{Z}}^*)_J = \emptyset$, then $R(\mathcal{DO}_{X_I}(\lambda)) = 0$.

2.3 Extremal cases

We characterize the extremal cases (see Definition 1.5) in the correspondence (2.3). We shall only deal with the invertible \mathcal{O} -modules. Given $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I^0$ and $\mu \in (\mathfrak{h}_{\mathbb{Z}}^*)_J^0$, we write for short (λ, μ) instead of $(\mathcal{O}_{X_I}(\lambda), \mathcal{O}_{X_J}(\mu))$.

Proposition 2.10. *The pair (λ, μ) is an extremal case if and only if $\mu = \lambda + \gamma_{I,J}$. This condition is also equivalent to the following system*

$$\begin{cases} \lambda(\alpha_i^\vee) = \mu(\alpha_i^\vee) = 0 & (i \in I \cap J), \\ \lambda(\alpha_i^\vee) = 0, \mu(\alpha_i^\vee) = \gamma_{I,J}(\alpha_i^\vee) & (i \in I \setminus J), \\ \lambda(\alpha_i^\vee) = -\gamma_{I,J}(\alpha_i^\vee), \mu(\alpha_i^\vee) = 0 & (i \in J \setminus I), \\ \mu(\alpha_i^\vee) - \lambda(\alpha_i^\vee) = \gamma_{I,J}(\alpha_i^\vee) & (i \in I_0 \setminus (I \cup J)). \end{cases} \quad (2.22)$$

Proof. The first statement is obvious by (2.5). Since $\Delta^+ \setminus \Delta_I$ and Δ_J are stable under the action of W_I and W_J respectively, we have $w(\gamma_{I,J}) = \gamma_{I,J}$ for any $w \in W_{I \cap J} = W_I \cap W_J$. In particular, we have

$$\gamma_{I,J} = s_i(\gamma) = \gamma_{I,J} - \gamma_{I,J}(\alpha_i^\vee)\alpha_i$$

for any $i \in I \cap J$. Hence we obtain

$$\gamma_{I,J}(\alpha_i^\vee) = 0 \quad \text{for any } i \in I \cap J.$$

Therefore, the relation $\mu = \lambda + \gamma_{I,J}$ is equivalent to the system (2.22). \square

By (2.22) we have the following

Corollary 2.11. *If \mathfrak{g} is semisimple and $I \cup J = I_0$, there exists a unique extremal case for (2.3).*

Proposition 2.12. *If (λ, μ) is an extremal case, then we have*

$$(\lambda + \rho)(\alpha^\vee) \begin{cases} < 0 & \text{for any } \alpha \in \Delta_J^+ \setminus \Delta_I, \\ > 0 & \text{for any } \alpha \in \Delta_I^+, \end{cases}$$

and $(w_J w_{I \cap J}) \circ \lambda = \mu$. In particular, we have $e \in \Gamma(\lambda)$ and $\ell(e) = m(e) = 0$.

Proof. Since μ and $\gamma_{I,J}$ are fixed by the action of W_J and $W_{I \cap J}$ respectively, We have

$$(w_J w_{I \cap J}) \circ \lambda = w_J w_{I \cap J}(\mu - \gamma_{I,J} + \rho) - \rho = \mu - w_J(\gamma_{I,J} - w_{I \cap J} \rho + w_J \rho).$$

By

$$w_{I \cap J} \rho - w_J \rho = (\rho - w_J \rho) - (\rho - w_{I \cap J} \rho) = \sum_{\alpha \in \Delta_J^+} \alpha - \sum_{\alpha \in \Delta_{I \cap J}^+} \alpha = \gamma_{I,J}$$

we obtain $(w_J w_{I \cap J}) \circ \lambda = \mu$. Hence by $w_J w_{I \cap J}(\Delta_J^+ \setminus \Delta_I) \subset \Delta_J^-$ and $\mu \in (\mathfrak{h}_{\mathbb{Z}}^*)_J^0$, we have

$$(\lambda + \rho)(\alpha^\vee) = (w_{I \cap J} w_J(\mu + \rho))(\alpha^\vee) = (\mu + \rho)(w_J w_{I \cap J} \alpha^\vee) < 0$$

for any $\alpha \in \Delta_J^+ \setminus \Delta_I$. Moreover, we have $(\lambda + \rho)(\alpha^\vee) > 0$ for any Δ_I^+ by (2.22). \square

By Proposition 1.6, if the pair (λ, μ) is an extremal case we get a nontrivial \mathcal{D}_{X_J} -linear morphism

$$\Phi : \mathcal{DO}_{X_J}(\mu) \rightarrow H^0(R(\mathcal{DO}_{X_I}(\lambda))). \quad (2.23)$$

Theorem 2.13. *Let (λ, μ) be an extremal case.*

- (i) *We have $H^p(R(\mathcal{DO}_{X_I}(\lambda))) = 0$ for any $p \neq 0$ if and only if $\ell(x) \geq m(x)$ for any $x \in \Gamma(\lambda)$.*
- (ii) *Assume that $H^p(R(\mathcal{DO}_{X_I}(\lambda))) = 0$ for any $p \neq 0$. Then Φ is an epimorphism if and only if $\ell(x) > m(x)$ for any $x \in \Gamma(\lambda) \setminus \{e\}$.*
- (iii) *Assume that $H^p(R(\mathcal{DO}_{X_I}(\lambda))) = 0$ for any $p \neq 0$. Then Φ is an isomorphism if and only if $\Gamma(\lambda) = \{e\}$.*

We need the following result in order to prove Theorem 2.13.

Lemma 2.14. *Let (λ, μ) be an extremal case, and let $x_1, x_2 \in \Gamma(\lambda)$. Set $y_k = y_{x_k}$ for $k = 1, 2$. If $L((y_1 x_1) \circ \lambda)$ appears as a subquotient of $M_J((y_2 x_2) \circ \lambda)$, then we have $\ell(x_2) - \ell(y_2) \leq \ell(x_1) - \ell(y_1)$.*

Proof. For $\xi \in \mathfrak{h}_{\mathbb{Z}}^*$ we set

$$\Delta_0^+(\xi) = \{\alpha \in \Delta^+ : (\xi + \rho)(\alpha^\vee) = 0\}, \quad W_0(\xi) = \{w \in W : w \circ \xi = \xi\}.$$

Take $\nu \in W \circ \lambda$ such that $(\nu + \rho)(\alpha^\vee) \geq 0$ for any $\alpha \in \Delta^+$, and let $w \in W$ such that $\lambda = w \circ \nu$. We can assume that $\ell(w) \leq \ell(x)$ for any $x \in W$ satisfying $\lambda = x \circ \nu$. Then w is the (unique) element of $wW_0(\nu)$ with minimal length.

Let us first show:

$$y_k x_k w \text{ is the element of } y_k x_k w W_0(\nu) \text{ with minimal length.} \quad (2.24)$$

It is sufficient to show $y_k x_k w \Delta_0^+(\nu) \subset \Delta^+$. Since w is the element of $wW_0(\nu)$ with minimal length, we have $w \Delta_0^+(\nu) \subset \Delta^+$, and hence $w \Delta_0^+(\nu) = \Delta_0^+(\lambda)$. By Proposition 2.12 we have $\Delta_0^+(\lambda) \subset \Delta^+ \setminus \Delta_I$. Hence by $W_I(\Delta^+ \setminus \Delta_I) = \Delta^+ \setminus \Delta_I$ we have $x_k \Delta_0^+(\lambda) \subset \Delta^+$. Thus $x_k \Delta_0^+(\lambda) = \Delta_0^+(x_k \circ \lambda)$. By $x_k \in \Gamma(\lambda)$ we have $\Delta_0^+(x_k \circ \lambda) \subset \Delta^+ \setminus \Delta_J$, and hence $y_k \Delta_0^+(x_k \circ \lambda) \subset \Delta^+$. The statement (2.24) is proved.

We next show

$$\ell(y_k x_k w) = \ell(w) + \ell(x_k) - \ell(y_k). \quad (2.25)$$

For any $\alpha \in \Delta_I^+$ we have

$$(\nu + \rho)(w^{-1} \alpha^\vee) = (\lambda + \rho)(\alpha^\vee) > 0,$$

and hence $w^{-1} \Delta_I^+ \subset \Delta^+$ by the choice of ν . Thus we have

$$w^{-1}(x_k^{-1} \Delta^+ \cap \Delta^-) = w^{-1}(x_k^{-1} \Delta_I^+ \cap \Delta_I^-) \subset w^{-1} \Delta_I^- \subset \Delta^-.$$

Hence $\ell(x_k w) = \ell(w) + \ell(x_k)$. Here, we have used the well-known fact that for $u, v \in W$ we have $\ell(uv) = \ell(u) + \ell(v)$ if and only if $u(v \Delta^+ \cap \Delta^-) \subset \Delta^-$. Similarly, we have

$$(\nu + \rho)(w^{-1} x_k^{-1} y_k^{-1} \alpha^\vee) = (y_k x_k (\lambda + \rho))(\alpha^\vee) > 0$$

for any $\alpha \in \Delta_J^+$ by the definition of y_k and hence $w^{-1} x_k^{-1} y_k^{-1} \Delta_J^+ \subset \Delta^+$. Thus we have

$$w^{-1} x_k^{-1} y_k^{-1} (y_k \Delta^+ \cap \Delta^-) = w^{-1} x_k^{-1} y_k^{-1} (y_k \Delta_J^+ \cap \Delta_J^-) \subset w^{-1} x_k^{-1} y_k^{-1} \Delta_J^- \subset \Delta^-.$$

Hence $\ell(x_k w) = \ell(y_k x_k w) + \ell(y_k)$. The statement (2.25) is proved.

Note that $L((y_1 x_1) \circ \lambda) = L((y_1 x_1 w) \circ \nu)$ and that $M_J((y_2 x_2) \circ \lambda)$ is a quotient of the ordinary Verma module $M((y_2 x_2 w) \circ \nu) = M_\emptyset((y_2 x_2 w) \circ \nu)$. Hence by our assumption and by (2.24) we obtain $y_1 x_1 w \geq y_2 x_2 w$ with respect to the standard partial order on W by a result of Bernstein-Gelfand-Gelfand [2] concerning the composition factors of Verma modules. In particular, we have $\ell(y_1 x_1 w) \geq \ell(y_2 x_2 w)$. Hence we obtain the desired result by (2.25). \square

Proof of Theorem 2.13. We shall use the notation in Proposition 2.5.

We first show the following.

$$\text{If } H^r(\mathcal{M}(k)^\bullet) = 0 \text{ for any } k \geq \ell, \text{ then } H^r(\mathcal{C}(k)^\bullet) = 0 \text{ for any } k \geq \ell. \quad (2.26)$$

Assume that there exists some $k \geq \ell$ such that $H^r(\mathcal{C}(k)^\bullet) \neq 0$. Let k_0 be the largest such k . Then we have exact sequences

$$H^{r-1}(\mathcal{M}(k_0 + 1)^\bullet) \rightarrow H^r(\mathcal{C}(k_0)^\bullet) \rightarrow 0, \quad (2.27)$$

$$H^{r-1}(\mathcal{C}(k)^\bullet) \rightarrow H^{r-1}(\mathcal{M}(k)^\bullet) \rightarrow H^{r-1}(\mathcal{M}(k+1)^\bullet) \rightarrow 0 \quad (k > k_0). \quad (2.28)$$

By $H^r(\mathcal{C}(k_0)^\bullet) \neq 0$ there exists some $x_1 \in \Gamma(\lambda)$ such that $\ell(x_1) - m(x_1) = -r$, $\ell(x_1) = k_0$ and $\mathcal{DO}_{X_J}((y_{x_1}x_1) \circ \lambda)$ is a direct summand of $H^r(\mathcal{C}(k_0)^\bullet)$. On the other hand by (2.27) and (2.28) any irreducible subquotient of $H^r(\mathcal{C}(k_0)^\bullet)$ is isomorphic to an irreducible subquotient of $H^{r-1}(\mathcal{C}(k)^\bullet)$ for some $k \geq k_0 + 1$. Moreover, $H^{r-1}(\mathcal{C}(k)^\bullet)$ is isomorphic to the direct sum of $\mathcal{DO}_{X_J}((y_{x_2}x_2) \circ \lambda)$ for $x_2 \in \Gamma(\lambda)$ such that $\ell(x_2) - m(x_2) = -(r-1)$, $\ell(x_2) = k$. By the category equivalence given in Proposition 1.9 we see that there exists some $x_2 \in \Gamma(\lambda)$ such that $\ell(x_2) - m(x_2) = -(r-1)$, $\ell(x_2) \geq k_0 + 1$, and that $L((y_{x_1}x_1) \circ \lambda)$ is isomorphic to an irreducible subquotient of $M_J((y_{x_2}x_2) \circ \lambda)$. Then by Lemma 2.14 we have

$$\ell(x_2) - \ell(y_{x_2}) \leq \ell(x_1) - \ell(y_{x_1}). \quad (2.29)$$

On the other hand we have

$$\ell(x_2) + \ell(y_{x_2}) = \ell(x_1) + \ell(y_{x_1}) + 1. \quad (2.30)$$

by Lemma 2.7. Hence we have $2\ell(x_2) \leq 2\ell(x_1) + 1$. Since $\ell(x_1)$ and $\ell(x_2)$ are integers, we obtain $\ell(x_2) \leq \ell(x_1)$. This is a contradiction. The statement (2.26) is proved.

Let us show (i). By Theorem 2.6(iv) we have $H^p(R(\mathcal{DO}_{X_I}(\lambda))) = 0$ for any $p \neq 0$ if $\ell(x) \geq m(x)$ for any $x \in \Gamma(\lambda)$. Assume that $H^p(R(\mathcal{DO}_{X_I}(\lambda))) = 0$ for any $p > 0$ and that $\ell(x) < m(x)$ for some $x \in \Gamma(\lambda)$. Then we have $H^p(\mathcal{M}(0)^\bullet) = 0$ for any $p > 0$ and $H^p(\mathcal{C}(k)^\bullet) \neq 0$ for some $p > 0$ and some $k \geq 0$. Let r be the largest positive integer such that $H^r(\mathcal{C}(k)^\bullet) \neq 0$ for some $k \geq 0$. Then we have an exact sequence

$$H^r(\mathcal{M}(k)^\bullet) \rightarrow H^r(\mathcal{M}(k+1)^\bullet) \rightarrow 0 \quad (k \geq 0).$$

Since $H^r(\mathcal{M}(0)^\bullet) = 0$, we see by induction on k that $H^r(\mathcal{M}(k)^\bullet) = 0$ for any $k \geq 0$. Hence by (2.26) we have $H^r(\mathcal{C}(k)^\bullet) = 0$ for any $k \geq 0$. This is a contradiction. The statement (i) is proved.

Let us show (ii). By (i) and the assumption we have $\ell(x) \geq m(x)$ for any $x \in \Gamma(\lambda)$; in other words $H^p(\mathcal{C}(k)^\bullet) = 0$ for any $p > 0$ and any $k \geq 0$. By Theorem 2.6(v) Φ is an epimorphism if $\ell(x) > m(x)$ for any $x \in \Gamma(\lambda) \setminus \{e\}$. Assume that Φ is an epimorphism. Since $\Phi : H^0(\mathcal{C}(0)^\bullet) \rightarrow H^0(\mathcal{M}(0)^\bullet)$ is an epimorphism, we have $H^0(\mathcal{M}(k)^\bullet) = 0$ for any $k > 0$ by the exact sequences

$$\begin{aligned} H^0(\mathcal{C}(0)^\bullet) &\rightarrow H^0(\mathcal{M}(0)^\bullet) \rightarrow H^0(\mathcal{M}(1)^\bullet) \rightarrow 0, \\ H^0(\mathcal{M}(k)^\bullet) &\rightarrow H^0(\mathcal{M}(k+1)^\bullet) \rightarrow 0 \end{aligned}$$

Hence by (2.26) we have $H^0(\mathcal{C}(k)^\bullet) = 0$ for any $k > 0$. It implies that $\ell(x) > m(x)$ for any $x \in \Gamma(\lambda) \setminus \{e\}$. The statement (ii) is proved.

Let us finally show (iii). By (i) and the assumption we have $H^p(\mathcal{C}(k)^\bullet) = 0$ for any $p > 0$ and any $k \geq 0$. By Theorem 2.6(v) Φ is an isomorphism if $\Gamma(\lambda) = \{e\}$.

Hence it is sufficient to show that $H^{-p}(\mathcal{C}(k)^\bullet) = 0$ for any $k > 0$ and any $p \geq 0$ if Φ is an isomorphism. Let us show it by induction on p . If $p = 0$, then we have $H^0(\mathcal{C}(k)^\bullet) = 0$ for any $k > 0$ by the proof of (ii). Assume that the statement is proved up to p . Consider the exact sequence

$$H^{-(p+1)}(\mathcal{M}(0)^\bullet) \rightarrow H^{-(p+1)}(\mathcal{M}(1)^\bullet) \rightarrow H^{-p}(\mathcal{C}(0)^\bullet) \rightarrow H^{-p}(\mathcal{M}(0)^\bullet).$$

We have $H^{-p}(\mathcal{C}(0)^\bullet) = 0$ for $p > 0$, and $\Phi : H^{-p}(\mathcal{C}(0)^\bullet) \rightarrow H^{-p}(\mathcal{M}(0)^\bullet)$ is an isomorphism for $p = 0$. Moreover, we have $H^{-(p+1)}(\mathcal{M}(0)^\bullet) = 0$ by Lemma 1.4. Hence we have $H^{-(p+1)}(\mathcal{M}(1)^\bullet) = 0$. Thus we obtain $H^{-(p+1)}(\mathcal{M}(k)^\bullet) = 0$ for any $k > 0$ by the exact sequence

$$H^{-(p+1)}(\mathcal{M}(k)^\bullet) \rightarrow H^{-(p+1)}(\mathcal{M}(k+1)^\bullet) \rightarrow H^{-p}(\mathcal{C}(k)^\bullet)$$

and the hypothesis of induction. Hence we have $H^{-(p+1)}(\mathcal{C}(k)^\bullet) = 0$ for any $k > 0$ by (2.26). The statement (iii) is proved. \square

By using Theorem 2.13 (i) and a case-by-case analysis we obtain the following.

Theorem 2.15. *Assume that G is a simple group of classical type and that $\sharp(I) = \sharp(J) = \sharp(I_0) - 1$. Let (λ, μ) be the extremal case. Then we have $H^p(R(\mathcal{DO}_{X_I}(\lambda))) = 0$ unless $p = 0$.*

Details of the proof is omitted. We do not know an example of an extremal case (λ, μ) satisfying $H^p(R(\mathcal{DO}_{X_I}(\lambda))) \neq 0$ for some $p > 0$.

Remark 2.16. Let (λ, μ) be an extremal case. For $x \in \Gamma$ and $\alpha \in \Delta_J^+ \setminus \Delta_I$ we have

$$(x(\lambda + \rho))(\alpha^\vee) = (\lambda + x\rho)(\alpha^\vee) = (\mu - \gamma_{I,J} + x\rho)(\alpha^\vee) = (x\rho - \gamma_{I,J})(\alpha^\vee),$$

and hence we have $H^p(R(\mathcal{DO}_{X_I}(\lambda))) = 0$ for any $p > 0$ if and only if

$$\begin{cases} \text{for } x \in \Gamma \text{ satisfying } (x\rho - \gamma_{I,J})(\alpha^\vee) \neq 0 \text{ for any } \alpha \in \Delta_J^+ \setminus \Delta_I \\ \text{we have } \sharp\{\alpha \in \Delta_J^+ \setminus \Delta_I : (x\rho - \gamma_{I,J})(\alpha^\vee) > 0\} \leq \ell(x). \end{cases} \quad (2.31)$$

In the next section we shall give conditions in order that Φ is an epimorphism and that Φ is an isomorphism under the assumption of Theorem 2.15. In particular, Φ is not necessarily an epimorphism nor a monomorphism. It seems to be an interesting problem to determine the kernel and the cokernel of Φ .

Remark 2.17. In Tanisaki [15] it is shown in certain cases that $\text{Ker}\Phi$ corresponds to the unique maximal proper submodule of $M_J(\mu)$ under the category equivalence given in Proposition 1.9.

3 The maximal parabolic case for classical simple groups

In this section we apply our results to the case where G is a classical simple group and P_I, P_J are maximal parabolic subgroups, and obtain results for the Radon transform $R(\mathcal{DO}_{X_I}(\lambda))$ with respect to the geometric correspondence

$$X_I \xleftarrow{f} X_{I \cap J} \xrightarrow{g} X_J$$

for $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I^0$. In this case we have

$$I = I_0 \setminus \{p\} \text{ and } J = I_0 \setminus \{q\} \text{ for some } p \neq q, \quad (3.1)$$

and $(\mathfrak{h}_{\mathbb{Z}}^*)_I^0 = \{r\varpi_p : r \in \mathbb{Z}\}$, where ϖ_k denotes the fundamental weight corresponding to $k \in I_0$.

We keep the standard notations of Bourbaki [4]. In particular, if G is of rank n , then $I_0 = \{1, 2, \dots, n\}$.

3.1 The case (A_n)

In this subsection we consider the case where $G = SL(V)$ for an $n+1$ -dimensional complex vector space V . By the symmetry of the Dynkin diagram we may (and shall) assume that $p > q$. We have the identifications:

$$\begin{aligned} X_I &= \{p\text{-dimensional subspace of } V\}, \\ X_J &= \{q\text{-dimensional subspace of } V\}, \\ X_{I \cap J} &= \{(U_1, U_2) \in X_I \times X_J : U_1 \supset U_2\}, \end{aligned}$$

and f, g are natural projections. The invertible \mathcal{O}_{X_I} -module $\mathcal{O}_{X_I}(\varpi_p)$ corresponds to the tautological line bundle whose fiber at $U \in X_I$ is $\bigwedge^p U$ (a subbundle of the product bundle $X_I \times \bigwedge^p V$), and we have $\mathcal{O}_{X_I}(r\varpi_p) = \mathcal{O}_{X_I}(\varpi_p)^{\otimes r}$. Hence in the standard notation of algebraic geometry we have $\mathcal{O}_{X_I}(r\varpi_p) = \mathcal{O}_{X_I}(-r)$.

For $k \in I_0 = \{1, \dots, n\}$ set

$$k_* = n + 1 - k, \quad k_+ = \max\{k, k_*\}, \quad k_- = \min\{k, k_*\}.$$

We first give consequences of Theorem 2.6. A weight $\lambda = \sum_{i=1}^{n+1} \lambda_i \varepsilon_i$ ($\lambda_i \in \mathbb{Z}$, $\sum_{i=1}^{n+1} \lambda_i = 0$) belongs to $(\mathfrak{h}_{\mathbb{Z}}^*)_J$ if and only if $\lambda_1 \geq \dots \geq \lambda_q$ and $\lambda_{q+1} \geq \dots \geq \lambda_{n+1}$. The Weyl group W is identified with the symmetric group S_{n+1} , and it acts on the weights by permutations of the components, i. e. $\sigma\lambda = \sum_{i=1}^{n+1} \lambda_i \varepsilon_{\sigma(i)}$ for any $\sigma \in W$. Then we have $W_I = S_p \times S_{p_*}$ and $W_J = S_q \times S_{q_*}$. We have

$$\begin{aligned} \varpi_p &= \frac{1}{n+1} [(n+1-p)(\varepsilon_1 + \dots + \varepsilon_p) - p(\varepsilon_{p+1} + \dots + \varepsilon_{n+1})] \\ &= \varepsilon_1 + \dots + \varepsilon_p + \text{const}(\varepsilon_1 + \dots + \varepsilon_{n+1}) \\ \rho &= \frac{1}{2} [n\varepsilon_1 + (n-2)\varepsilon_2 + \dots + (-n)\varepsilon_{n+1}] \\ &= -\varepsilon_2 - \dots - n\varepsilon_{n+1} + \text{const}(\varepsilon_1 + \dots + \varepsilon_{n+1}), \end{aligned}$$

and therefore we get

$$r\varpi_p + \rho = r\varepsilon_1 + (-1 + r)\varepsilon_2 + \cdots + (-(p-1) + r)\varepsilon_p - p\varepsilon_{p+1} - \cdots - n\varepsilon_{n+1} \\ + \text{const}(\varepsilon_1 + \cdots + \varepsilon_{n+1}).$$

By the assumption $q < p$ the set $\Gamma(r\varpi_p)$ consists of $(\sigma, \tau) \in S_p \times S_{p^*}$ satisfying

$$\begin{cases} \tau = e, \\ \sigma^{-1}(1) < \cdots < \sigma^{-1}(q), \\ \sigma^{-1}(q+1) < \cdots < \sigma^{-1}(p), \\ \{\sigma^{-1}(q+1), \dots, \sigma^{-1}(p)\} \cap \{p+r+1, \dots, n+r+1\} = \emptyset, \end{cases}$$

and we have

$$\ell((\sigma, e)) = \sharp\{(a, b) : 1 \leq a \leq q, q+1 \leq b \leq p, \sigma^{-1}(a) > \sigma^{-1}(b)\}, \\ m((\sigma, e)) = \sharp\{(b, c) : q+1 \leq b \leq p, p+1 \leq c \leq n+1, \sigma^{-1}(b) < r+c\}.$$

Hence by Theorem 2.6 we obtain the following results.

Proposition 3.1. (i) Assume $q < p_-$. Then we have $R(\mathcal{DO}_{X_I}(-a\varpi_p)) = 0$ if $q+1 \leq a \leq q_* - 1$.

(ii) Assume $q \leq p_-$. Then we have

$$R(\mathcal{DO}_{X_I}(-q_*\varpi_p)) = \mathcal{DO}_{X_J}(-p_*\varpi_q), \\ R(\mathcal{DO}_{X_I}(-q\varpi_p)) = \mathcal{DO}_{X_J}(-p\varpi_q)[-(p-q)(p_*-q)].$$

(iii) $H^k(R(\mathcal{DO}_{X_I}(-a\varpi_p))) = 0$ for any $k \neq 0$ if $a \geq q_*$.

Let us consider the extremal case. By

$$\gamma_{I,J} = p_* \sum_{i=q+1}^p \varepsilon_i - (p-q) \sum_{i=p+1}^{n+1} \varepsilon_i.$$

and (2.22) the extremal case is given by $(-q_*\varpi_p, -p_*\varpi_q)$. By Theorem 2.13 we obtain the following.

Proposition 3.2. We have $H^k(R(\mathcal{DO}_{X_I}(-q_*\varpi_p))) = 0$ for any $k \neq 0$, and there exists a canonical nontrivial epimorphism

$$\Phi : \mathcal{DO}_{X_J}(-p_*\varpi_q) \rightarrow H^0(R(\mathcal{DO}_{X_I}(-q_*\varpi_p))).$$

Moreover, Φ is an isomorphism if and only if $p^* \geq q$.

Remark 3.3. In the situation of Proposition 3.2 it is proved in [15] that for $p^* \leq q$ the kernel of Φ is the maximal proper G -stable submodule of $\mathcal{DO}_{X_J}(-p_*\varpi_q)$.

In the rest of this subsection we assume that $q < p_-$ and give application to topological problems. By Proposition 3.1 we have

$$R(\mathcal{DO}_{X_I}(-a\varpi_p)) \simeq \begin{cases} \mathcal{DO}_{X_J}(-p_*\varpi_q) & \text{for } a = q_*, \\ 0 & \text{for } q+1 \leq a \leq q_*-1, \\ \mathcal{DO}_{X_J}(-p\varpi_q)[-l_{pq}] & \text{for } a = q, \end{cases} \quad (3.2)$$

where $l_{pq} = (p-q)(p_*-q)$. Thus by Proposition 1.7 we have the following.

Proposition 3.4. *For any $F \in \mathbf{D}^b(\mathbb{C}_{X_{J,\text{an}}})$ and $q+1 \leq a \leq q_*-1$ we have*

$$\begin{aligned} \mathrm{R}\Gamma(X_{I,\text{an}}; r(F) \otimes \mathcal{O}_{X_I}(a\varpi_p)_{\text{an}}) &= 0, \\ \mathrm{RHom}(r(F), \mathcal{O}_{X_I}(a\varpi_p)_{\text{an}}) &= 0, \end{aligned}$$

and for $(a, b, c, d) = (q_*, p_*, (p-q)p_*, pp_* - qq_* - q(p-q))$ or $(a, b, c, d) = (q, p, q(p-q), -q(p-q))$ we have

$$\begin{aligned} \mathrm{R}\Gamma(X_{I,\text{an}}; r(F) \otimes \mathcal{O}_{X_I}(a\varpi_p)_{\text{an}}) &\simeq \mathrm{R}\Gamma(X_{J,\text{an}}; F \otimes \mathcal{O}_{X_J}(b\varpi_q)_{\text{an}})[-c], \\ \mathrm{RHom}(r(F), \mathcal{O}_{X_I}(a\varpi_p)_{\text{an}}) &\simeq \mathrm{RHom}(F, \mathcal{O}_{X_J}(b\varpi_q)_{\text{an}})[-d]. \end{aligned}$$

Let us treat some particular cases. In the following we set $N = qq_*$.

(1) Let $y_o \in X_J$, and set $F = \mathbb{C}_{\{y_o\}}$. Since $g^{-1}(y_o) \rightarrow X_{I,y_o}$ is a closed embedding, one has

$$r(F) \simeq \mathbb{C}_{X_{I,y_o,\text{an}}}, \quad (3.3)$$

where $X_{I,y_o} = fg^{-1}(\{y_o\}) = \{x \in X_I : y_o \subset x\}$ (identified with the Grassmannian of $(p-q)$ -subspaces of V/y_o). By Proposition 3.4 and (3.3) we obtain the following.

Proposition 3.5. *For any $q+1 \leq a \leq q_*-1$ we have*

$$\mathrm{R}\Gamma(X_{I,y_o,\text{an}}; \mathcal{O}_{X_I}(a\varpi_p)_{\text{an}}) \simeq 0, \quad \mathrm{R}\Gamma_{X_{I,y_o,\text{an}}}(X_I; \mathcal{O}_{X_I}(a\varpi_p)_{\text{an}}) \simeq 0,$$

and for $(a, c, d) = (q_*, (p-q)p_*, pp_* - qq_* + p_*q)$ or $(a, c, d) = (q, q(p-q), p_*q)$ we have

$$H^c(X_{I,y_o,\text{an}}; \mathcal{O}_{X_I}(a\varpi_p)_{\text{an}}) \simeq \mathbb{C}\{z\}, \quad H_{X_{I,y_o,\text{an}}}^d(X_I; \mathcal{O}_{X_I}(a\varpi_p)_{\text{an}}) \simeq \mathcal{B}_{0|\mathbb{C}^N}^\infty$$

where $\mathbb{C}\{z\}$ (resp. $\mathcal{B}_{0|\mathbb{C}^N}^\infty$) is the ring of convergent power series in $z = (z_1, \dots, z_N) \in \mathbb{C}^N$ (resp. the ring of hyperfunctions in \mathbb{C}^N along $\{0\}$ of infinite order), and all other cohomology groups vanish.

Namely, one identifies $\mathrm{R}\Gamma(X_{J,\mathrm{an}}; \mathbb{C}_{y_0} \otimes \mathcal{O}_{X_J}(b\varpi_q)_{\mathrm{an}}) \simeq \mathrm{R}\Gamma(\{0\}; \mathcal{O}_{\mathbb{C}_{\mathrm{an}}^N}) = \mathbb{C}\{z\}$ and $\mathrm{RHom}(\mathbb{C}_{y_0}; \mathcal{O}_{X_J}(b\varpi_q)_{\mathrm{an}}) \simeq \mathrm{R}\Gamma_{\{0\}}(\mathbb{C}_{\mathrm{an}}^N; \mathcal{O}_{\mathbb{C}_{\mathrm{an}}^N}) = \mathcal{B}_{0|\mathbb{C}^N}^\infty[-N]$.

(2) Let z_0 be a q_* -subspace of V , $E_{z_0} = \{y \in X_J : y \cap z_0 = 0\} \simeq \mathbb{C}^N$ and set $F = \mathbb{C}_{E_{z_0},\mathrm{an}}$. One has

$$r(F) \simeq \mathbb{C}_{\widehat{E}_{z_0},\mathrm{an}}[-2q(p-q)], \quad (3.4)$$

where $\widehat{E}_{z_0} = fg^{-1}(E_{z_0}) = \{x \in X_I : \dim(x \cap z_0) = p-q\}$ (i.e. the p -dimensional subspaces of V in generic position w.r.t. z_0). Namely, the map $\tilde{f} = (f|_{g^{-1}(E_{z_0})})_{\mathrm{an}} : (g^{-1}(E_{z_0}))_{\mathrm{an}} \rightarrow \widehat{E}_{z_0,\mathrm{an}}$ is a complex vector bundle of rank $q(p-q)$ (the fiber over $x \in \widehat{E}_{z_0}$ is $S_{E_{z_0},x} = \{y \in E_{z_0} : y \subset x\} \simeq \mathbb{C}^{q(p-q)}$); hence there is a morphism of functors $R\tilde{f}_* \tilde{f}^{-1}[2q(p-q)] \rightarrow \mathrm{id}_{\mathbf{D}^b(\mathbb{C}_{\widehat{E}_{z_0},\mathrm{an}})}$ defining a natural morphism $r(F) = Rf_{\mathrm{an}}! \mathbb{C}_{(g^{-1}(E_{z_0}))_{\mathrm{an}}} \rightarrow \mathbb{C}_{\widehat{E}_{z_0},\mathrm{an}}[-2q(p-q)]$, which is an isomorphism since, by (1.4), one has $r(F)_x \simeq \mathbb{C}[-2q(p-q)]$ (for $x \in \widehat{E}_{z_0,\mathrm{an}}$) and $= 0$ (otherwise).

By Proposition 3.4 and (3.4) we obtain the following.

Proposition 3.6. *For any $q+1 \leq a \leq q_*-1$ we have*

$$\mathrm{R}\Gamma_c(\widehat{E}_{z_0,\mathrm{an}}; \mathcal{O}_{X_I}(a\varpi_p)_{\mathrm{an}}) \simeq 0, \quad \mathrm{R}\Gamma(\widehat{E}_{z_0,\mathrm{an}}; \mathcal{O}_{X_I}(a\varpi_p)_{\mathrm{an}}) \simeq 0,$$

and for $(a, c, d) = (q_*, p(p_* - q) + q^2, p_*(p - q))$ or $(a, c, d) = (q, p_*q, q(p - q))$ we have

$$\begin{aligned} H_c^c(\widehat{E}_{z_0,\mathrm{an}}; \mathcal{O}_{X_I}(a\varpi_p)_{\mathrm{an}}) &\simeq H_c^N(E_{z_0,\mathrm{an}}; \mathcal{O}_{E_{z_0,\mathrm{an}}}), \\ H^d(\widehat{E}_{z_0,\mathrm{an}}; \mathcal{O}_{X_I}(a\varpi_p)_{\mathrm{an}}) &\simeq \Gamma(E_{z_0,\mathrm{an}}; \mathcal{O}_{E_{z_0,\mathrm{an}}}) \end{aligned}$$

where $H_c^N(E_{z_0,\mathrm{an}}; \mathcal{O}_{E_{z_0,\mathrm{an}}}) \simeq \Gamma(E_{z_0,\mathrm{an}}; \Omega_{E_{z_0,\mathrm{an}}})'$ (resp. $\Gamma(E_{z_0,\mathrm{an}}; \mathcal{O}_{E_{z_0,\mathrm{an}}})$) are Martineau's analytic functionals (resp. the entire functions) in $E_{z_0,\mathrm{an}} \simeq \mathbb{C}^N$, and all other cohomology groups vanish.

Namely, one has $\mathrm{R}\Gamma(X_{J,\mathrm{an}}; \mathbb{C}_{E_{z_0,\mathrm{an}}} \otimes \mathcal{O}_{X_J}(b\varpi_q)_{\mathrm{an}}) \simeq H_c^N(E_{z_0,\mathrm{an}}; \mathcal{O}_{E_{z_0,\mathrm{an}}})[-N]$ and $\mathrm{RHom}(\mathbb{C}_{E_{z_0,\mathrm{an}}}; \mathcal{O}_{X_J}(b\varpi_q)_{\mathrm{an}}) \simeq \Gamma(E_{z_0,\mathrm{an}}; \mathcal{O}_{E_{z_0,\mathrm{an}}})$.

3.2 The case (B_n)

In this subsection we consider the case where G is (the universal covering group of) $SO(V)$ for an $2n+1$ -dimensional complex vector space V equipped with a non-degenerate symmetric bilinear form $(,) : V \times V \rightarrow \mathbb{C}$. Then we have the identifications:

$$\begin{aligned} X_I &= \{p\text{-dimensional subspace } U \text{ of } V \text{ such that } (U, U) = 0\}, \\ X_J &= \{q\text{-dimensional subspace } U \text{ of } V \text{ such that } (U, U) = 0\}, \\ X_{I \cap J} &= \begin{cases} \{(U_1, U_2) \in X_I \times X_J : U_1 \subset U_2\} & (p < q) \\ \{(U_1, U_2) \in X_I \times X_J : U_1 \supset U_2\} & (p > q), \end{cases} \end{aligned}$$

and f, g are natural projections. The invertible \mathcal{O}_{X_I} -module $\mathcal{O}_{X_I}(\varpi_p)$ corresponds to the tautological line bundle whose fiber at $U \in X_I$ is $\bigwedge^p U$.

By Theorem 2.6 we have the following.

Proposition 3.7. (i) We have $R(\mathcal{D}\mathcal{O}_{X_I}(-a\varpi_p)) = 0$ in the following cases:

$$\begin{cases} 2n - p - q < a < q & \text{if } p < q \leq n, \\ \min(n - p, q) < a < n - q & \text{if } q < p < n, \\ n - p + q < a < \max(n, 2n - p - q) & \text{if } q < p < n, \\ 2q < a < 2(n - q) & \text{if } p = n, \end{cases}$$

(ii) We have $R(\mathcal{D}\mathcal{O}_{X_I}(-a\varpi_p)) = \mathcal{D}\mathcal{O}_{X_I}(-b\varpi_q)[-c]$ in the following cases:

$$(a, b, c) = \begin{cases} (q, p, 0) & (p < q < n, 2n - 2p - q \leq 0), \\ (2n - p - q, 2(n - q), c_1) & (p < q \leq n, 2n - 2p - q \leq 0), \\ (n, 2p, 0) & (q = n, n - 2p \leq 0), \\ (2n - p - q, 2n - p - q, 0) & (q < p < n, 2n - 2p - q \geq 0), \\ (q, p, c_2) & (q < p < n, 2n - 2p - q \geq 0), \end{cases}$$

where

$$c_1 = \frac{(q - p)(3p + 3q - 4n - 1)}{2}, \quad c_2 = \frac{(p - q)(4n + 1 - 3p - 3q)}{2}.$$

By Theorem 2.13 we have the following.

Proposition 3.8. Let

$$(r, s) = \begin{cases} (q, p) & \text{if } 1 \leq p < q \leq n - 1, \\ (2n - p - q, 2n - p - q) & \text{if } 1 \leq q < p \leq n - 1, \\ (2(n - q), n - q) & \text{if } p = n, 1 \leq q \leq n - 1, \\ (n, 2p) & \text{if } 1 \leq p \leq n - 1, q = n. \end{cases}$$

Then we have $H^k(R(\mathcal{D}\mathcal{O}_{X_I}(-r\varpi_p))) = 0$ for any $k \neq 0$, and there exists a canonical nontrivial morphism

$$\Phi : \mathcal{D}\mathcal{O}_{X_I}(-s\varpi_q) \rightarrow H^0(R(\mathcal{D}\mathcal{O}_{X_I}(-r\varpi_p))).$$

Moreover, Φ is an epimorphism if and only if we have either

$$(a) \quad p < q \leq n,$$

$$(b) \quad q < p < n \text{ and } 2n - 2p - q \geq 0,$$

and an isomorphism if and only if we have either

$$(a) \quad p < q \leq n \text{ and } 2n - 2p - q \leq 0,$$

$$(b) \quad q < p < n \text{ and } 2n - 2p - q \geq 0.$$

3.3 The case (C_n)

In this subsection we consider the case where $G = Sp(V)$ for an $2n$ -dimensional complex vector space V equipped with a non-degenerate anti-symmetric bilinear form $(,) : V \times V \rightarrow \mathbb{C}$. Then we have the identifications:

$$\begin{aligned} X_I &= \{p\text{-dimensional subspace } U \text{ of } V \text{ such that } (U, U) = 0\}, \\ X_J &= \{q\text{-dimensional subspace } U \text{ of } V \text{ such that } (U, U) = 0\}, \\ X_{I \cap J} &= \begin{cases} \{(U_1, U_2) \in X_I \times X_J : U_1 \subset U_2\} & (p < q) \\ \{(U_1, U_2) \in X_I \times X_J : U_1 \supset U_2\} & (p > q), \end{cases} \end{aligned}$$

and f, g are natural projections. The invertible \mathcal{O}_{X_I} -module $\mathcal{O}_{X_I}(\varpi_p)$ corresponds to the tautological line bundle whose fiber at $U \in X_I$ is $\bigwedge^p U$.

By Theorem 2.6 we have the following.

Proposition 3.9. (i) *We have $R(\mathcal{D}\mathcal{O}_{X_I}(-a\varpi_p)) = 0$ in the following cases:*

$$\begin{cases} 2n - p - q + 1 < a < q & \text{if } p < q, \\ q < a < 2n - p - q + 1 & \text{if } q < p. \end{cases}$$

(ii) *We have $R(\mathcal{D}\mathcal{O}_{X_I}(-a\varpi_p)) = \mathcal{D}\mathcal{O}_{X_J}(-b\varpi_q)[-c]$ in the following cases:*

$$(a, b, c) = \begin{cases} (q, p, 0) & (p < q \leq n, 2n - 2p - q + 1 \leq 0), \\ (2n - p - q + 1, 2n - p - q + 1, c_1) & (p < q \leq n, 2n - 2p - q + 1 \leq 0), \\ (2n - p - q + 1, 2n - p - q + 1, 0) & (q < p \leq n, 2n - 2p - q + 1 \geq 0), \\ (q, p, c_2) & (q < p \leq n, 2n - 2p - q + 1 \geq 0), \end{cases}$$

where

$$c_1 = \frac{(q-p)(3p+3q-4n-1)}{2}, \quad c_2 = \frac{(p-q)(4n+1-3p-3q)}{2}.$$

By Theorem 2.13 we have the following.

Proposition 3.10. *Let*

$$(r, s) = \begin{cases} (q, p) & \text{if } 1 \leq p < q \leq n, \\ (2n - p - q + 1, 2n - p - q + 1) & \text{if } 1 \leq q < p \leq n. \end{cases}$$

Then we have $H^k(R(\mathcal{D}\mathcal{O}_{X_I}(-r\varpi_p))) = 0$ for any $k \neq 0$, and there exists a canonical nontrivial morphism

$$\Phi : \mathcal{D}\mathcal{O}_{X_J}(-s\varpi_q) \rightarrow H^0(R(\mathcal{D}\mathcal{O}_{X_I}(-r\varpi_p))).$$

Moreover, Φ is an epimorphism if and only if we have either

- (a) $p < q < n$ and $n - p - q \geq 0$,
- (b) $p < q \leq n$ and $2n - 2p - q + 1 \leq 0$,
- (c) $q < p \leq n$,

and an isomorphism if and only if we have either

- (a) $p < q \leq n$ and $2n - 2p - q + 1 \leq 0$,
- (b) $q < p \leq n$ and $2n - 2p - q + 1 \geq 0$.

Remark 3.11. In the situation of Proposition 3.10 it is proved in [15] that $\text{Ker}\Phi$ is the maximal proper G -stable submodule of $\mathcal{DO}_{X_J}(-s\varpi_q)$ if $q = n$ and $2p \leq n - 1$.

3.4 The case (D_n)

In this subsection we consider the case where G is (the universal covering group of) $SO(V)$ for an $2n$ -dimensional complex vector space V equipped with a non-degenerate symmetric bilinear form $(\cdot, \cdot) : V \times V \rightarrow \mathbb{C}$.

For $1 \leq k \leq n$ set

$$X(k) = \{k\text{-dimensional subspace } U \text{ of } V \text{ such that } (U, U) = 0\}.$$

Then $X(k)$ is connected for $1 \leq k \leq n - 1$, and $X(n)$ has two connected components, say $X_1(n)$ and $X_2(n)$. Then we have the identification:

$$\begin{aligned} X(k) &= X_{I_0 \setminus \{k\}} \quad (1 \leq k \leq n - 2), \\ X(n - 1) &= X_{I_0 \setminus \{n-1, n\}}, \\ X_1(n) &= X_{I_0 \setminus \{n\}}, \\ X_2(n) &= X_{I_0 \setminus \{n-1\}}. \end{aligned}$$

If $\{p, q\} \neq \{n - 1, n\}$, then

$$X_{I \cap J} = \begin{cases} \{(U_1, U_2) \in X_I \times X_J : U_1 \subset U_2\} & (p < q) \\ \{(U_1, U_2) \in X_I \times X_J : U_1 \supset U_2\} & (p > q), \end{cases}$$

and if $p = n - 1$ and $q = n$, then f (resp. g) assigns $U \in X_{I \cap J} = X(n - 1)$ to the unique $U' \in X_I = X_2(n)$ (resp. $U' \in X_J = X_1(n)$) such that $U \subset U'$. The invertible \mathcal{O}_{X_I} -module $\mathcal{O}_{X_I}(\varpi_p)$ corresponds to the tautological line bundle whose fiber at $U \in X_I$ is $\bigwedge^k U$ where $k = p$ for $1 \leq k \leq n - 2$ and $k = n$ for $p \in \{n - 1, n\}$.

By Theorem 2.6 we have the following.

Proposition 3.12. (i) We have $R(\mathcal{DO}_{X_I}(-a\varpi_p)) = 0$ in the following cases:

$$\begin{cases} 2n - p - q - 1 < a < q & \text{if } p < q \leq n - 2, \\ q < a < 2n - p - q - 1 & \text{if } q < p \leq n - 2 \\ 2q < a < 2(n - q - 1) & \text{if } p \in \{n - 1, n\}, 1 \leq q \leq n - 2, \\ n - p - 1 < a < n & \text{if } 1 \leq p \leq n - 2, q \in \{n - 1, n\}, \\ a = n - 1 & \text{if } \{p, q\} = \{n - 1, n\} \text{ and } n \text{ is even.} \end{cases}$$

(ii) We have $R(\mathcal{DO}_{X_I}(-a\varpi_p)) = \mathcal{DO}_{X_J}(-b\varpi_q)[-c]$ in the following cases:

$$(a, b, c) = \begin{cases} (q, p, 0) & (p < q \leq n - 2, 2n - 2p - q - 1 \leq 0), \\ (2n - p - q - 1, 2n - p - q - 1, c_1) & (p < q \leq n - 2, 2n - 2p - q - 1 \leq 0), \\ (n, 2p, 0) & (p \leq n - 2, q \in \{n - 1, n\}, n - 2p - 1 \leq 0), \\ (n - p - 1, 2(n - p - 1), c_2) & (p \leq n - 2, q \in \{n - 1, n\}, n - 2p - 1 \leq 0), \\ (2n - p - q - 1, 2n - p - q - 1, 0) & (q < p \leq n - 2, 2n - 2p - q - 1 \geq 0), \\ (q, p, c_3) & (q < p \leq n - 2, 2n - 2p - q - 1 \geq 0), \\ (n, n - 2, 0) & (\{p, q\} = \{n - 1, n\}, n : \text{odd}), \\ (n - 1, n - 1, 0) & (\{p, q\} = \{n - 1, n\}, n : \text{odd}), \\ (n - 2, n, 0) & (\{p, q\} = \{n - 1, n\}, n : \text{odd}), \end{cases}$$

where

$$c_1 = \frac{(q - p)(3p + 3q - 4n + 1)}{2}, \quad c_2 = \frac{(n - p)(3p - n + 1)}{2}, \\ c_3 = \frac{(p - q)(4n - 3p - 3q - 1)}{2}.$$

By Theorem 2.13 we have the following.

Proposition 3.13. Let

$$(r, s) = \begin{cases} (q, p) & \text{if } 1 \leq p < q \leq n - 2, \\ (2n - p - q - 1, 2n - p - q - 1) & \text{if } 1 \leq q < p \leq n - 2, \\ (2(n - q - 1), n - q - 1) & \text{if } p \in \{n - 1, n\}, 1 \leq q \leq n - 2, \\ (n, 2p) & \text{if } 1 \leq p \leq n - 2, q \in \{n - 1, n\}, \\ (n, n - 2) & \text{if } \{p, q\} = \{n - 1, n\}. \end{cases}$$

Then we have $H^k(R(\mathcal{DO}_{X_I}(-r\varpi_p))) = 0$ for any $k \neq 0$, and there exists a canonical nontrivial epimorphism

$$\Phi : \mathcal{DO}_{X_J}(-s\varpi_q) \rightarrow H^0(R(\mathcal{DO}_{X_I}(-r\varpi_p))).$$

Moreover, Φ is an isomorphism if and only if we have either

- (a) $p < q < n - 1$ and $2n - 2p - q - 1 \leq 0$,
- (b) $q < p < n - 1$ and $2n - 2p - q - 1 \geq 0$,
- (c) $p < n - 1$, $q \in \{n - 1, n\}$ and $n - 2p - 1 \leq 0$,
- (c) $\{p, q\} = \{n - 1, n\}$ and n is odd.

Remark 3.14. In the situation of Proposition 3.13 it is proved in [15] that $\text{Ker}\Phi$ is the maximal proper G -stable submodule of $\mathcal{DO}_{X_J}(-s\varpi_q)$ if $q \in \{n - 1, n\}$, $2p \leq n - 2$ and if $q = 1$, $p \in \{n - 1, n\}$.

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